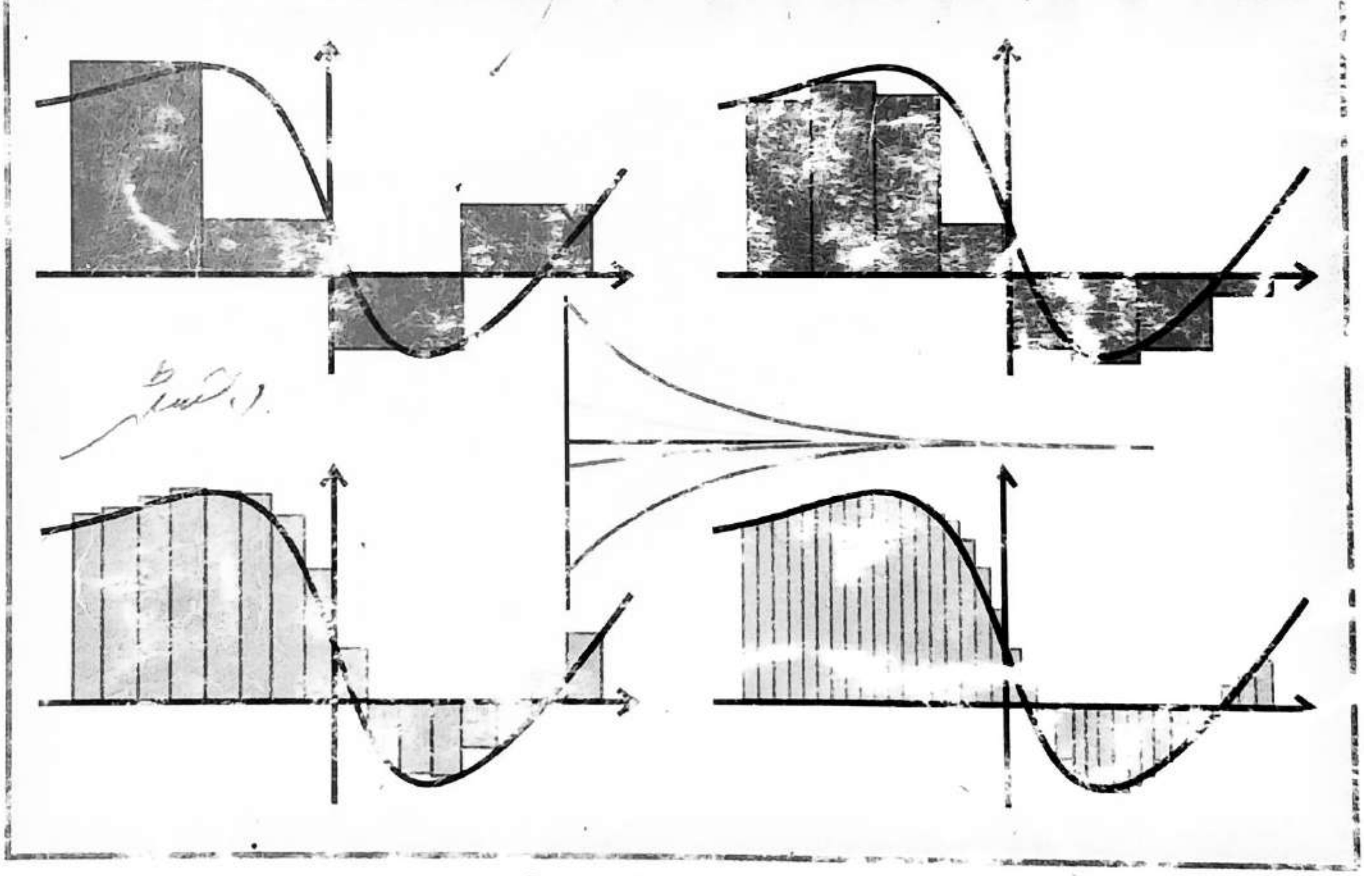
TEXTBOOK OF
APPLIED MATHEMATICS-II

MATH-233

Diploma of Associate Engineer
According To New Course



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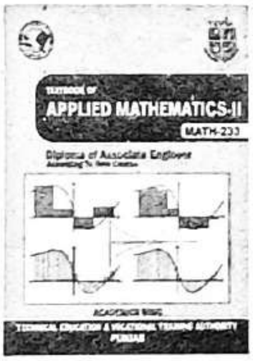
Applied Mathematics - II
MATH-233



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A Textbook of

APPLIED MATHEMATICS-II

Math–233

for

Second year
Diploma of Associate Engineer
(DAE)

Including:
Objective Type
and
Short Questions

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Preface

We are thankful to Almighty ALLAH Who gave us an opportunity to write the book , named 'Applied Mathematics-II ' as Textbook of Diploma of Associate Engineer (DAE), intending to cover the new syllabus for the first year students

Throughout the book, emphasis is on correct methods of computation, transposition of formulae, logical layout of solutions, neatness and clarity of arrangement of material , systematic use of all the normal mathematical and other tables.

Topics covered include Functions & limits, Derivatives and its Applications, Statistics and Probability, Integration & its Applications, Fourier series and Laplace Transformation.

Normally students face difficulty in solving complicated problems because they do not make a systematic attempt. We have attempted to help the students to overcome the difficulty by providing detailed instructions for an orderly approach. Difficult procedures and types of problems appearing in the exercise are illustrated by carefully explained examples. In the presentation of these illustrated examples, we have avoided unnecessary explanations. It is hoped that this book will help to give students a good foundation in old and new techniques.

Students are reminded that in order to acquire a proper understanding of the subject and its application, it is necessary to learn a number of sound basic rules and methods. No scientific or engineering subject can be fully comprehended and satisfactorily studied without a sound mathematical background.

We made every effort to make the book valuable both for students and teachers, however we shall gratefully welcome to receive any suggestion for the further improvement of the book.

Authors.

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Chapter 1

Functions and Limits

1.1 Introduction:

The temperature at which water boils depends on the elevation above sea level. The boiling point drops as you ascend. Also interest paid on a cash investment depends on the length of time for which the investment is held. In both the cases the value of one variable quantity which might be y, depends on the another variable quantity which we might be x. Since value of y is completely determined by the value of x, we say that y is a function of x.

1.2 Basic concepts :

1-Quantity :

Anything that can be measured or counted is called a quantity . length, mass, volume , time , age are examples of quantities. These are of two types :

i -Constants :

A constant is a quantity which retains the same values throughout a problem .i.e., the quantity to which we can not assign any other value. e.g., the sum of angles of a triangle is a constant.

ii -Variables :

Those quantities which go on changing during a given interval of time and consequently assume an unlimited numbers of different values.

Kinds of variables :

(a) Independent variables :

The variables which varies independently. i.e., the variable in which the changes(in values) produce corresponding changes in other variable.

(b) Dependent variables ;

That variable whose value depends upon the value assigned to the other variable. e.g., volume of a sphere is given by $\mathbf{V = \frac{4}{3}\pi r^3}$ varies only when r (radius) varies. So , r is an independent variable. Where V is dependent variable .

1.3 Function :

If two variables have some relation with each other, then dependent variable is called function of independent variable. i.e.,

If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y, then we say that ***"y is a function of x"***

The usual notation to express that one variable y is a function of another variable x is.

$y = f(x)$ or $y = g(x)$ etc.

A function with two or more independent variables is represented by the symbol f(x,y) or g(x, y, z) etc.

Example I:

Area of a circle is given as $A = \pi r^2$.

Since $A = \pi r^2$ varies as r varies ,so Area of the circle is a function of 'r'.

The equation $A = \pi r^2$ is a rule that tells how to calculate unique output of A for each possible input value of radius r.

Note: i. The set of all possible input values of the radius is called the ***domain of the function.***

ii. The set of all out put values of the area is the ***range of the functions.***

Example 2:

Let $y = x^2 + 3x + 2$ be an equation. Since for each value of x there is always a corresponding value of y, therefore y is said to be a function of x.

There are some examples of function.

$y = \text{six}^2\, x$

$$y = \frac{\ln x}{x}$$

$y = x e^x$

1.4 Evaluation of function :

A function y = f(x) can be evaluated by substituting that particular value in place of x in the given function.

Example 3:

Suppose that the function f is defined for all real numbers x by the formula
$f(x) = 2(x - 1) + 3$

Evaluate f at the input value, 0, 2, x + 2 and f(f(2))

Solution:

In each case we substitute the given input value for x into the formula for f.

$f(0) = 2(0 - 1) + 3 = -2 + 3 = 1$

$f(2) = 2(2 - 1) + 3 = 2 + 3 = 5$

$f(x + 2) = 2(x + 2 - 1) + 3 = 2x + 5$

$f(f(2)) = 2(f(2) - 1) + 3 = 2(5 - 1) + 3 = 11$

Example 4:

If $f(x) = 2x^2 + 4x + 9$, find the value of $\dfrac{f(3) - f(1)}{f(-1) + f(0)}$

Solution : Since $f(x) = 2x^2 + 4x + 9$

$f(0) = 2(0) + 4(0) + 9 = 9$

$$\begin{aligned} f(1) &= 2(1)^2 + 4(1) + 9 &&= 15 \\ f(-1) &= 2(-1)^2 + 4(-1) + 9 &&= 7 \\ f(3) &= 2(3)^2 + 4(3) + 9 &&= 39 \end{aligned}$$

$$\therefore \quad \frac{f(3) - f(1)}{f(-1) + f(0)} = \frac{39-15}{7+9} = \frac{24}{16} = \frac{3}{2}$$

1.5 Algebra of Function:

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions.

If f and g are function then for every value of x that belongs to the domains for both f and g, we define functions f + g, f – g and fg as

$$(f + g)(x) = f(g) + g(x)$$

$$(f - g)(x) = f(x) - g(f)$$

$$(fg)(x) = f(x) \,.\, g(x)$$

At the point at which $g(x) \neq 0$, we can, define the function f/g by the formula.

$$\left[\frac{f}{g}\right](x) = \frac{f(x)}{g(x)} \text{ where } g(x) \neq 0$$

Function can also be multiplied by a real constant say c. Then function cf is defined for all x in the domain of f by.

$$f(c\,x) = c\,f(x)$$

Example 5:

If f(x) = ln x, Prove that

i. $f(p\,q) = f(p) + f(q)$ ii. $F\left(\frac{p}{q}\right) = f(p) - f(q)$

Solution:

$$f(x) = \text{In } x$$

$$\text{i.} \quad \begin{aligned} f(p\,q) &= \text{In}(\,p\,q) \\ &= \text{In } p + \ln q \\ &= f(p) + f(q) \end{aligned}$$

$$\text{ii.} \quad \begin{aligned} f\left[\frac{p}{q}\right] &= \text{In}\left[\frac{p}{q}\right] \\ &= \text{In } p - \text{In } q \\ &= f(p) - f(q) \end{aligned}$$

Example 6:

If f(x) = Sin x + Cos x, Show that $f(x + \pi) = - f(x)$

Solution:

Since $f(x) = \text{Sin } x + \text{Cos } x$

$$\begin{aligned} f(x+\pi) &= \text{Sin}(x+\pi) + \text{Cos}(x+\pi) \\ &= \sin x \cos \pi + \cos x \sin \pi + \cos x \cos \pi - \sin x \sin \pi \\ &= \text{Sin } x(-1) + \text{Cos } x(0) + \text{Cos } x(-1) - \text{Sin } x(0) \\ &= -\text{Sin } x - \text{Cos } x \\ &= -(\text{Sin } x + \text{Cos } x) \\ f(x+\pi) &= -f(x) \end{aligned}$$

1.6 Types of Functions and Their Classification

1.6.1 Explicit and Implicit Functions

Explicit Functions :

A function which is given in terms of the independent variable is called an Explicit function.

It can be expressed as $y = f(x)$,

where y is dependent variable and x is an independent variable. i.e.,

i $y = x^2 + 3x - 8$

ii. $y = \sqrt{x^2 + 2x}$

iii. $y = 1 + \sin 2x$

Implicit Functions :

A function which can be written in terms of both dependent and independent variable is called an Implicit function.

It can be expressed as $f(x, y) = 0$

i.e., i. $x^3 + 3x^2 + 4xy^2 + y^3 + 3 = 0$

ii. $x^2 \sin y + y \cos x = 7$

iii. $\sin xy + y^3 = x$

Note : Some implicit functions can be reduced in Explicit functions e.g.

(i) $x^2 y + x y = a^3$ can be reduced as $y = \dfrac{a^3}{x^2 + x}$

(ii) $y - xy - x^2 + 4 = 0$ can be reduced as $y = \dfrac{x^2 - 4}{1 - x}$

1.6.2 Algebraic and Transcendental Functions

Algebraic functions are functions $y = f(x)$ satisfying an equation of the form $p_o(x)\, y^n + p_1(x) y^{n-1} + \ldots\ldots + p_{n-1}(x)\, y + p_n(x) = 0$

Where $p_o(x), p_1(x), \ldots\ldots, p_n(x)$ are polynomial in x

i.e., i. $y = x^3 + 3x^2 + \dfrac{2}{x} + \dfrac{4}{x^2} + 5$

ii. $y = a x^2 + b x + c$

are examples of **algebraic functions.**

On the other hand **transcendental functions** are the functions which are not algebraic.

i.e.
- i. Trigonometric functions
  $\sin x, \cos x, \tan x, \cot x$ $\sec x$ and $\operatorname{cosec} x$
- ii. Inverse trigonometric functions
  $\sin^{-1}x, \cos^{-1}x, \tan^{-1}x, \cot^{-1}x, \sec^{-1}x$ & $\operatorname{cosec}^{-1}x$
- iii. Logarithmic functions
  $\log x$ or $\ln x$
- iv. Exponential functions
  a^x and e^x

are examples of transcendental functions.

1.6.3 Even and Odd Functions

Even function : A function f(x) is said to be an **Even function** of x

if for $x = -x \Rightarrow f(-x) = f(x)$

For example $f(x) = x^2$ and $f(x) = \cos x$ are even functions of x

As $f(-x) = (-x)^2 = x^2 = f(x)$

and $f(-x) = \cos(-x) = \cos x = f(x)$

Odd function : A function f(x) is said to be an **Odd function** of x

if for $x = -x \Rightarrow f(-x) = -f(x)$

For example $f(x) = x^3$ and $f(x) = \sin x$ are odd functions of x

As $f(-x) = (-x)^3 = -x^3 = -f(x)$

and $f(-x) = \sin(-x) = -\sin x = -f(x)$

Example 7:

Determine which functions are even, odd or neither.

a. $f(x) = 2x^3 - 9x$

b. $f(x) = 2x^4 - 3x^2 - 1$

c. $f(x) = 4x^2 - 7x + 6$

Solution:

a. $\therefore$
$$\begin{aligned} f(-x) &= 2(-x)^3 - 9(-x) \\ &= -2x^3 + 9x \\ &= -(2x^3 - 9x) \\ &= -f(x) \end{aligned}$$

So f is odd

b. $\therefore$
$$\begin{aligned} f(-x) &= 2(-x)^4 - 3(-x)^2 - 1 \\ &= 2x^4 - 3x^2 - 1 \\ &= f(x) \end{aligned}$$

So f is even

c. $\therefore$
$$\begin{aligned} f(-x) &= 4f(-x)^2 - 7(-x) + 6 \\ &= 4x^2 + 7x + 6 \neq f(x) \\ &\neq -f(x) \end{aligned}$$

The result is neither f(x) nor –f(x) for all values of x . So f is neither even nor odd.

Note: cos x , sec x are Even functions of x and
sin x , tan x , cot x , cosec x are Odd functions of x

Exercise 1.1

Q.1: (i) If $f(x) = \dfrac{x^2-3}{x+4}$, find f(–3) (ii) If $f(x) = 3x^2 - 5x + 7$, find f(4)

Q.2: Find the value of $f\left(\dfrac{1}{x}\right)$, if $f(x) = \dfrac{1}{x^2+4}$

Q.3: If $f(x) = 3x^3 + 2x^2 - x + 4$, prove that $2f(3) = 25f(1)$

Q.4: Given $f(x) = 3x^3 - ax^2 + bx + 1$, if $f(2) = -3$ and $f(-1) = 0$, Find the values of a and b

Q.5: (a) If $f(x) = a^x$, show that

(i) $f(-p) = \dfrac{1}{f(p)}$ (ii) $f(x+y) = f(x)\,f(y)$

(b) If $f(x) = \log x$, prove that $f(x^a) = a\,f(x)$

Q.6: If $f(p) = p + \dfrac{1}{p}$. Prove the following results:

(i) $f(-p) = -f(p)$ (ii) $f\left(\dfrac{1}{p}\right) = f(p)$

Q.7: If f(x) = tan x show that $f(x+y) = \dfrac{f(x)+f(y)}{1-f(x)\,f(y)}$

Q.8: If $f(t) = \dfrac{t^4+t^2+1}{t^2}$, show that $f\left(\dfrac{1}{t}\right) = f(t)$

Q.9: If $f(x) = \dfrac{x-1}{x+1}$, show that $\dfrac{f(x)-f(y)}{1+f(x)\,f(y)} = \dfrac{x-y}{1+xy}$.

Q.10: Prove that f[f(x)] = x for the function $f(x) = \dfrac{x+1}{x-1}$

Q.11: Are the following functions even odd or neither?

(i) $f(x) = 4x^3 - 2x + 6$ (ii) $f(x) = \dfrac{x}{x^2+1}$

(iii) $f(x) = 2x^4 - 3x^2 - 1$ (iv) $f(x) = x\sqrt{x^2-1}$

Q.12: (i) Show that $\frac{e^x+1}{e^x-1}$ is an odd function of x.

(ii) show that $x \cdot \frac{a^x+1}{a^x-1}$ is an Even function of x.

Q.13 : If $f(x) = \log \frac{1-x}{1+x}$ prove that $f(x) + f(y) = f\left(\frac{x+y}{1+xy}\right)$

$$\frac{e^{-x}+1}{e^{-x}-1} = \frac{\frac{1}{e^x}+1}{\frac{1}{e^x}-1} = \frac{\frac{1+e^x}{e^x}}{\frac{1-e^x}{e^x}} = \frac{1+e^x}{1-e^x} = -\left(\frac{e^x+1}{e^x-1}\right)$$

Answers

Q.1(i) 6 (ii) 35 **Q.2** $\frac{x^2}{1+4x^2}$ **Q.4** a = - 2 , b = - 6

Q.11 (i) Neither (ii) Odd (iii) Even (iv) Odd

LIMITS

1.7 Introduction:

The concept of limit of a function is one of the fundamental ideas that distinguishes calculus from algebra and trigonometry. The concept of limit is the foundation of almost all the mathematical analysis. A good understanding of limit will help to explain many theories of calculus.

1.8 Value and Limit of a Function:

There are two ways of studying a function. One is to find the value of y, the dependent variable for various values of x, the independent variable by actual substitution in the functional relation between the two variables. Another method to study the behavior of the dependent variable in the neighborhood of various values of the independent variable. The former method is finding the ***value of the function*** and the later method, which gives us more detailed information about the function than the former, is finding the ***limit of the function.***

1.8.1 Limit of a Variable: ($x \to a$)

If a variable `x' approaches a constant `a' such that the absolute value of the difference x - a becomes less than 'ϵ', any +ve number, however small, then `a' is called the limit of x and is symbolically written as $x \to a$ and read as "x approaches a" or " x tends to a".

For Example:

Suppose that x has the values $a+\frac{1}{10}, a+\frac{1}{10^2}, \ldots\ldots, a+\frac{1}{10^n}$

Since the values $\frac{1}{10}, \frac{1}{10^2}, \ldots\ldots, \frac{1}{10^n} \to 0$ when $n \to \infty$

Hence $x \to a$

1.8.2 x approaches to ∞ or − ∞

If a variable go on increasing and the values are greater than any real number, however large, then we say that $x \to \infty$. Similarly when the values taken by x becomes smaller and smaller and are less than every negative number, then we say that $x \to -\infty$

1.8.3 Limit of a Function:

If a function f(x) is defined for all values of x in some interval and if x approaches a given number 'a' in that interval then f(x) may approaches some definite number 'ℓ' which is called the limit of f(x) as $x \to a$ and is symbolically written as:

$$\lim_{x \to a} f(x) = l$$

i.e. when x is very near to 'a' then f(x) is very near to ℓ.

1.9 Theorem on Limits of functions:

Let f(x) and g (x) are function and k be a real number, then

1. Constant $\lim_{x \to a} k = k$
2. Sum rule $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
3. Difference rule $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
4. Product rule $\lim_{x \to a} [f(x)\, g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$
5. Constant multiple rule $\lim_{x \to a} kg(x) = k \lim_{x \to a} g(x)$
6. Quotient rule $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$; where $g(x) \neq 0$
7. Power rule If n is integer then $\lim_{x \to a} [f(x)]^n = \left(\lim_{x \to a} f(x)\right)^n$

1.10 Evaluation of limit of a function:

For evaluation of the limits of algebraic functions, there are two most important methods (a) direction method (b) indirect method

(a) Direct Method:

In cases, where values of the function does not assume any indeterminate form by putting the value of x directly to which it tends, in the function. The limit of the function is obtained.

ote : The indeterminate values of the function are of the form

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$$\frac{0}{0}, \frac{\infty}{\infty}, 0.\infty,\ \infty-\infty,\ 0^{\infty},\ 1^{\infty},\ \infty^{0},\ \infty\times\infty$$

Example 1: Evaluate $\lim\limits_{x\to 3}(x^2+2x)$ (Direct Method)

Solution: Since the limit is not indeterminate form for x = 3, so put directly $\lim\limits_{x\to 3}(x^2+2x) = 3^2+2(3) = 9+6 = 15$ Ans

(b) Indirect method:

Suppose as $x \to a$, the given limit is of the form 0/0 i.e., indeterminate

Then we use the following methods :

(1) Factorization **(2) Substitution** **(3) Rationalization**

(1) Factorization method :

1st Step: Make factors of numerator and denominator.

2nd Step: Cancel the common factor $x \to a$ (called disturbing factor) from numerator and denominator.

3rd Step: Put x = a in the remaining function which is the required limit of the function.

Example 2: Evaluate $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x^2-x}$

Solution: Given $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x^2-x}$ is of ($\dfrac{0}{0}$ form) for x = 1.

Factorizing the numerator and denominator we get

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x} = \lim_{x\to 1}\frac{(x-1)(x+2)}{x(x-1)}$$

$$= \lim_{x\to 1}\frac{x+2}{x}$$

$$= \frac{1+2}{1} = 3 \text{ Ans}$$

$x^2 - x + 2x - 2$

$x(x-1) + 2(x-1)$

$(x+2)(x-1)$

(2) Substitution Method(h- method):

Sometimes it becomes difficult to make the factors, then apply substitution, as

1st Step: Put x = a + h in the given function.

2nd Step: As $x \to a$, $h \to 0$ and simplify.

3rd Step: Cancel h (the disturbing factor) from numerator and denominator.

4th Step: Put h = 0 in the reduced function which gives the required limit.

Example 3: Evaluate $\lim\limits_{x\to 3}\frac{x^3-4x^2+9}{x^2-x-6}$

Solution: Given limit is of indeterminate form $\frac{0}{0}$ for x = 3

$-6x^2$

Put $x = 3 + h$, when $x \to 3$, $h \to 0$

$$\therefore \lim_{x\to 3}\frac{x^3-4x^2+9}{x^2-x-6} = \lim_{h\to 0}\frac{(3+h)^3-4(3+h)^2+9}{(3+h)^2-(3+h)-6}$$

$27h + 9h^2$

$$= \lim_{h\to 0}\frac{27+h^3+9h(3+h)-4(9+6h+h^2)+9}{9+6h+h^2-3-h-6}$$

$$= \lim_{h\to 0}\frac{27+h^3+27h+9h^2-36-24h-4h^2+9}{h^2+5h}$$

$$= \lim_{h\to 0}\frac{h^3+5h^2+3h}{h^2+5h}$$

$$= \lim_{h\to 0}\frac{h(h^2+5h+3)}{h(h+5)}, \text{ cancel h}$$

$$= \lim_{h\to 0}\frac{h^2+5h+3}{h+5} = \frac{0+0+3}{0+5} \text{ (putting h = 0)}$$

$$= \frac{3}{5} \text{ Ans}$$

(3) Method of Rationalization:

In the case of irrational functions, if limit of f(x) is of the form $\frac{0}{0}$ i.e., indeterminate, proceed as under:

1st Step: Rationalize the numerator or denominator as the case may be.

2nd Step: Cancel the common factor (disturbing factor) from numerator and denominator i.e., $x - a$ if $x \to a$

3rd Step: Put x = a in the reduced function, which is the required value of the given limit.

Example 4: Evaluate the limits:

(i) $\lim\limits_{x \to 0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x}$ (ii) $\lim\limits_{x \to 0} \dfrac{\sqrt{1-x}-\sqrt{1+x}}{x}$

Solution:(i) Given limit $\lim\limits_{x \to 0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x}$ is of ($\dfrac{0}{0}$ form) for x = 0.

Rationalize the numerator.

$$\therefore \lim_{x \to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{x \to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} \cdot \frac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}+\sqrt{x}}$$

$$= \lim_{x \to 0} \frac{2+x-2}{x(\sqrt{2+x}+\sqrt{x})}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{2+x}+\sqrt{2}} = \frac{1}{\sqrt{2+0}+\sqrt{2}}$$

$$\therefore \lim_{x \to 0} \frac{\sqrt{2+x}+\sqrt{2}}{x} = \frac{1}{2\sqrt{2}} \text{ Ans}$$

Solution:(ii) Given $\lim\limits_{x \to 0} \dfrac{\sqrt{1-x}-\sqrt{1+x}}{x}$; $\dfrac{0}{0}$ form for x = 0

Rationalize the numerator

$$\therefore \lim_{x \to 0} \frac{\sqrt{1-x}-\sqrt{1+x}}{x} = \lim_{x \to 0} \frac{\sqrt{1-x}-\sqrt{1+x}}{x} \times \frac{\sqrt{1-x}+\sqrt{1+x}}{\sqrt{1-x}+\sqrt{1+x}}$$

$$= \lim_{x \to 0} \frac{1-x-1-x}{x(\sqrt{1-x}+\sqrt{1+x})}$$

$$= \lim_{x \to 0} \frac{-2x}{x(\sqrt{1-x}+\sqrt{1+x})}$$

$$= \lim_{x \to 0} \frac{-2}{\sqrt{1-x}+\sqrt{1+x}}$$

$$= \frac{-2}{\sqrt{1-0}+\sqrt{1+0}} = \frac{-2}{1+1} = \frac{-2}{2}$$

$$\therefore \lim_{x \to 0} \frac{\sqrt{1-x}-\sqrt{1+x}}{x} = -1 \text{ Ans}$$

1.11 Limit when x tends to infinity

Consider a function f(x) defined for large positive (or negative) value of x as increases indefinitely in the positive (or negative) direction. If the value of f(x) approaches a number 'b' then the limit of f(x) as x increases (or decreases) indefinitely is equal to b, denoted as

$$\lim_{x \to +\infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b$$

Note : $\lim\limits_{x \to +\infty} \dfrac{1}{x} = 0, \lim\limits_{x \to -\infty} \dfrac{1}{x} = 0, \lim\limits_{x \to \infty} \dfrac{1}{x} = 0$

1.12 Evaluation of Limit when x tends to ∞ :

When $x \to \infty$ and the limit takes the form $\frac{\infty}{\infty}$, we can use two methods.

Method - I:

Divide numerator and denominator by the highest power of x

Method – II:

Put $x = \frac{I}{y}$, then when $x \to \infty, y \to 0$

Example 5:

Method – I : Find $\lim\limits_{x \to \infty} \dfrac{2x^2 - 3x + 7}{3x^2 + 5x - 1}$

Divide numerator and Denominator by highest power of x i.e., x^2

$$= \lim_{x \to \infty} \frac{2 - \frac{3}{x} + \frac{7}{x^2}}{3 + \frac{5}{x} + \frac{1}{x^2}}$$

$$= \frac{\lim\limits_{x \to \infty}\left(2 - \frac{3}{x} + \frac{7}{x^2}\right)}{\lim\limits_{x \to \infty}\left(3 - \frac{5}{x} + \frac{7}{x^2}\right)} = \frac{2 - 0 + 0}{3 + 0 - 0} = \frac{2}{3} \qquad \text{Ans.}$$

Method – II:

Put $x = \frac{I}{y}$

When $x \to \infty, \quad y \to 0$

So,

$$\lim_{x\to\infty}\frac{2x^2-3x+7}{3x^2+5x-1}=\lim_{y\to 0}\frac{\frac{2}{y^2}-\frac{3}{y}+7}{\frac{3}{y^2}+\frac{5}{y}-1}$$

$$=\lim_{y\to 0}\frac{2-3y+7y^2}{3+5y-y^2}$$

$$=\frac{2-3(0)+7(0)}{3+5(0)-0}=\frac{2}{3}\quad\text{Ans.}$$

$$\frac{\frac{2-3y+7y}{y^2}}{\frac{3+5y-y^2}{y^2}}$$

$$\frac{2-3y+7y^2}{3+5y-y^2}$$

Example 6:

Evaluate $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2-1}}{2x+1}$

Solution:

We first divide numerator and denominator by x and get

$$\lim_{x\to\infty}\frac{\sqrt{x^2-1}}{2x+1}=\lim_{x\to\infty}\frac{\sqrt{1-\frac{1}{x^2}}}{2+\frac{1}{x}}$$

$$=\frac{\lim\limits_{x\to\infty}\sqrt{1-\frac{1}{x^2}}}{\lim\limits_{x\to\infty}\left(2+\frac{1}{x}\right)}==\frac{\sqrt{1-0}}{2+0}=\frac{1}{2}\quad\text{Ans.}$$

Exercise 1.2

Q.1: Evaluate the following limits

(i) $\lim\limits_{x\to 1}(x^3-3x+5)$ (ii) $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{\sqrt{x}+1}$ (iii) $\lim\limits_{x\to -2}\dfrac{x^2}{x+1}$

(iv) $\lim\limits_{x\to 3}\sqrt{25-x^2}$ (v) $\lim\limits_{x\to 0}\dfrac{4x+5}{x}$ (vi) $\lim\limits_{x\to 0}\dfrac{2x^2+5x+7}{x}$

Q.2: Evaluate the following limit

(i) $\lim\limits_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$

(ii) $\lim\limits_{x \to 0} \frac{\sqrt{4+x} - 2}{x}$

(iii) $\lim\limits_{x \to 0} \frac{x}{\sqrt{4+x} - 2}$

(iv) $\lim\limits_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$

(v) $\lim\limits_{h \to 0} \frac{(1-h)^2 - 1}{h}$

(vi) $\lim\limits_{h \to 0} \left[\frac{1}{h}\left(\frac{1}{\sqrt{1-h}} - 1\right)\right]$

Q.3 : Evaluate the following limits.

(i) $\lim\limits_{x \to 1} \frac{x - 1}{\sqrt{x+3} - 2}$

(ii) $\lim\limits_{x \to 1} \frac{\sqrt{x} - 3}{x - 9}$

(vi) $\lim\limits_{x \to 2} \frac{\sqrt{x+7} - 3}{x - 2}$

(iv) $\lim\limits_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$

Q.4: Evaluate the following limits.

(i) $\lim\limits_{x \to 4} \frac{x^2 - 16}{x - 4}$

(ii) $\lim\limits_{x \to 1} \frac{x^2 - 1}{x - 1}$

(iii) $\lim\limits_{x \to 1} \frac{x^3 - 1}{x^2 - 1}$

(iv) $\lim\limits_{x \to 2} \frac{x^2 - x - 2}{x^2 - 3x + 2}$

(v) $\lim\limits_{x \to 0} \frac{x^2 - 3x + 2}{x^2 - 4x + 4}$

(vi) $\lim\limits_{x \to 3} \frac{x^2 - 5x + 6}{x - 3}$

(vii) $\lim\limits_{x \to -3} \frac{x^2 - x - 12}{x^2 + 4x + 3}$

Q.5: Calculate the following limits.

(i). $\lim\limits_{x \to \infty} \frac{x^3 + 8x^2 + 6}{4x^3 + 7x^2 - 5x}$

(ii). $\lim\limits_{x \to \infty} \frac{2x^3 - 3x + 7}{6x - 7x^2 - 9x^3}$

(iii). $\lim\limits_{x \to \infty} \frac{3x^2 - 5x + 1}{x^2 + x + 2}$

(iv). $\lim\limits_{x \to \infty} \frac{x^2 + 1}{2x^3 - x}$

(v). $\lim\limits_{x \to \infty} \frac{\sqrt{x^2 - 1}}{3x + 1}$

(vi). $\lim\limits_{x \to \infty} (\sqrt{x^2 + 1} - x)$

Answers

Q.1: i. 3 ii. Zero iii. – 4 iv. 4
v. ∞ vi. ∞

Q.2: i $\frac{1}{2}$ ii. 1/4 iii. 4 iv. $\frac{1}{2\sqrt{x}}$
v. - 2 vi $-\frac{1}{2}$

Q.3: i 4 ii $\frac{1}{4}$ iii $\frac{1}{6}$ iv $\frac{1}{2\sqrt{a}}$

Q.4: i 8 ii 2 iii $\frac{3}{2}$ iv 3
v $\frac{1}{2}$ vi 1 vii $\frac{7}{2}$

Q.5: i. $\frac{1}{4}$ ii. $-\frac{2}{9}$ iii. 3 iv. Zero v. $\frac{1}{3}$ vi. 0

1.13. Some Important Limits:

1.13.1 Prove that

$$\lim_{x \to 0} \frac{x^n - a^n}{x - a} = na^{n-1}$$

Proof:

If we substitute x = a, the function is of the form $\frac{0}{0}$ and is not defined.

Let x = a + h and limit h tends to zero

$$\frac{x^n - a^n}{x - a} = \frac{(a + h)^n - a^n}{(a + h) - a}$$

Using binomial expansion

$$= \frac{a^n + na^{n-1}h + \frac{n(n-1)}{2!}a^{n-2}h^2 + \ldots\ldots\ldots\ldots\ldots - a^n}{a + h - a}$$

$$= \frac{na^{n-1}h + \frac{n(n-1)}{2!}a^{n-2}h^2 + \ldots\ldots\ldots\ldots}{h}$$

$$= \frac{nh\left\{a^{n-1} + \frac{n-1}{2!}a^{n-2}h + \ldots\ldots\right\}}{h}$$

$$= n\left\{a^{n-1} + \frac{n-1}{2!}a^{n-2}h + \ldots\ldots\right\}$$

$$\lim_{x \to 0} \frac{x^n - a^n}{x - a} = \lim_{x \to 0} n\left\{a^{n-1} + \frac{n-1}{2}a^{n-2}h + \ldots\ldots\right\}$$

$$= n\{a^{n-1} + 0 + 0 + \ldots\ldots\ldots\ldots\}$$

$$\lim_{x \to 0} \frac{x^n - a^n}{x - a} = na^{n-1}$$

1.13.2 Prove that:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \text{, where } \theta \text{ is in radian.}$$

Proof:

Let r be the radius of a circle with center at 0 such that m $\angle AOB = \theta$.

Where θ is in radian

Draw the tangent at A. Produce OB such that It meets the tangent at T. Also join A and B as shown.

From the figure, in right angled triangle OAT.

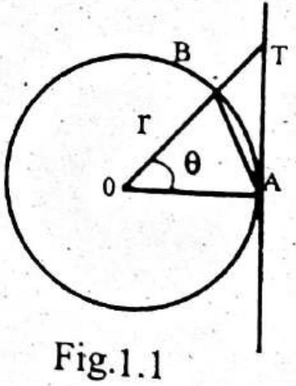

Fig.1.1

$$\frac{AT}{OA} = \tan\theta$$

$$\Rightarrow AT = OA\tan\theta \qquad \because \qquad OA = r$$

$$AT = r\tan\theta$$

Also Area of triangle OAB $= \frac{1}{2}r^2\sin\theta$

Area of sector OAB $= \frac{1}{2}r^2\theta$

Area of triangle OAT $= \frac{1}{2}\,OA \times AT = \frac{1}{2}r^2\tan\theta$

Now from the figure it is obvious that

Area of Δ AOB < Area of sector AOB < Area of Δ OAT

$$\frac{1}{2}r^2\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2\tan\theta$$

$\text{Sin}\,\theta < \theta < \tan\theta$

Dividing the inequality by sin θ.

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

Or $\quad 1 > \dfrac{\sin\theta}{\theta} > \cos\theta$

When $\theta \to 0$, $\cos\theta \to 1$

$1 > \lim\limits_{\theta\to 0} \dfrac{\text{Sin}\,\theta}{\theta} > 1$ (By using Sandwitch theorem)

Hence , $\lim\limits_{\theta\to 0} \dfrac{\text{Sin}\,\theta}{\theta} = 1$

1.11.3 Prove that:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.7185$$

Solution:

Expand using binomial series

$$\left(1+\frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\,\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\,\frac{1}{n^3} + \ldots\ldots\infty$$

$$= 1 + 1 + \frac{1}{2!}\,n^2\left(1-\frac{1}{n}\right)\frac{1}{n^2} + \frac{1}{3!}\,n^3\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{1}{n^3} + \ldots\ldots\infty$$

$$= 2 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots\ldots\ldots\ldots\ldots\ldots\infty$$

Now

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left[2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\right]+\ldots\ldots\infty$$

When $n\to\infty$ all $\dfrac{1}{n}, \dfrac{2}{n}, \dfrac{3}{n}, \ldots\ldots\ldots\to 0$

$$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots\ldots\ldots$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.712 = e$$

Example 1: **Evaluate the following limits**

i. $\lim\limits_{x\to 0} \dfrac{1-\cos\theta}{\theta^2}$ ii. $\lim\limits_{x\to 0} \dfrac{\sin px}{qx}$ iii $\lim\limits_{x\to \pi/2} \dfrac{\sec x}{\tan x}$

iv. $\lim\limits_{x\to 0} (1+3x)^{1/x}$

Solution:i

$$\lim_{x\to 0} \frac{1-\cos\theta}{\theta^2} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} \times \frac{1+\cos\theta}{1+\cos\theta}$$

$$= \lim_{\theta\to 0} \frac{1-\cos^2\theta}{\theta^2(1+\cos\theta)} = \lim_{\theta\to 0} \frac{\sin^2\theta}{\theta^2(1+\cos\theta)}$$

$$= \lim_{\theta\to 0}\left[\frac{\sin^2\theta}{\theta^2} \times \frac{1}{1+\cos\theta}\right]$$

$$= \left[\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\right]^2 \left[\lim_{\theta\to 0} \frac{1}{1+\cos\theta}\right]$$

$$= [1]^2\left[\frac{1}{1+1}\right] = \frac{1}{2}$$

ii. $\lim\limits_{x\to 0} \dfrac{\sin px}{qx}$

Solution:

$$\frac{\sin px}{qx} = \frac{\sin px}{px} \cdot \frac{p}{q}$$

$$\therefore \lim_{x\to 0} \frac{\sin px}{qx} = \frac{p}{q} \lim_{x\to 0} \frac{\sin px}{px}$$

$$= \frac{p}{q} 1 = \frac{p}{q}$$

iii $\lim\limits_{x\to \pi/2} \dfrac{\sec x}{\tan x}$

Solution:

$$\lim_{x\to \pi/2} \frac{\sec x}{\tan x} = \lim_{x\to \pi/2} \frac{1/\cos x}{\sin x/\cos x} = \lim_{x\to \pi/2} \frac{1}{\sin x}$$

$$= \frac{1}{\sin \pi/2} = \frac{1}{1} = 1$$

iv. $\lim\limits_{x\to 0} (1+3x)^{1/x}$

Solution:

$$\lim_{x\to 0}(1+3x)^{1/x} = \lim_{x\to 0}\left[(1+3x)^{1/3x}\right]^3$$

$$\text{put } 3x = \frac{1}{n}, \text{ as } x \to 0, n\to\infty$$

$$\therefore \lim_{x\to 0}\left[(1+3x)^{1/3x}\right]^3 = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^3 = e^3$$

Exercise 1.3

Q.1: Evaluate the following limits.

(i) $\lim_{\theta\to 0}\dfrac{\sin 7\theta}{\theta}$ (ii) $\lim_{x\to 0}\dfrac{\sin^2 x}{x^3}$

(iii) $\lim_{x\to 0}\dfrac{\tan x}{x}$ (iv) $\lim_{q\to 0}\dfrac{\tan pq}{q}$

(v) $\lim_{\theta\to 0}\dfrac{1-\cos\theta}{\theta\sin\theta}$ (vi) $\lim_{x\to 0}\dfrac{1-\cos 2x}{x^2}$

(vii) $\lim_{x\to 0}\dfrac{1-\cos x}{\sin^2 x}$ (viii) $\lim_{x\to 0}\dfrac{1-\sin x}{\cos^2 x}$

(ix) $\lim_{\theta\to 0}\dfrac{1-\cos p\theta}{1-\cos q\theta}$ (x) $\lim_{\theta\to 0}\dfrac{\tan\theta-\sin\theta}{\sin^3\theta}$

(xi) $\lim_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$ (xii) $\lim_{x\to 0}\dfrac{\operatorname{cosec} x-\cot x}{x}$

Q.2: Evaluate the following limits:

(i). $\lim_{x\to 0}(1+x)^{1/x}$ (ii). $\lim_{x\to\infty}\left(1-\dfrac{1}{x}\right)^x$

(iii). $\lim_{x\to\infty}\left(\dfrac{x}{1+x}\right)^x$ (iv). $\lim_{x\to\infty}\left(1+\dfrac{2}{x}\right)^x$

(v). $\lim_{x\to 0}(1+2x)^{1/x}$

Q.3: Evaluate the following limits.

(i) $\lim_{h\to 0}\dfrac{(1+h)^{\frac{3}{2}}-1}{h}$ (ii) $\lim_{x\to 1}\dfrac{x^{\frac{3}{2}}-1}{x-1}$ (iii) $\lim_{x\to a}\dfrac{x^n-a^n}{x^m-a^m}$

Answers

Q.1: i 7 ii. ∞ iii. 1 iv. P v. $\frac{1}{2}$
vi. 2 vii. $\frac{1}{2}$ viii. 1 ix. $\frac{p^2}{q^2}$ x $\frac{1}{2}$
xi. $\frac{1}{2}$ xii. $\frac{1}{2}$

Q.2: i. e ii. 1/e iii. 1/e iv. e^2
v. e^2

Q.3: i $\frac{3}{2}$ ii $\frac{3}{2}$ iii $\frac{n}{m}\,a^{n-m}$

Short Questions

Write the short answers of the following:

Q.1: If $f(x) = 3x^2 - 7x + 4$, then find $f\left(\frac{1}{x}\right)$.

Q.2: If $f(x) = 2x\sqrt{1 - x^2}$, find $f(\sin\theta)$

Q.3: If $f(x) = \frac{2x}{1 + x^2}$, Find $f(\tan A)$.

Q.4: If $f(x) = \frac{1}{1 - x}$, Then find $f[f(5)]$

Q.5: Show that the function $f(x) = x^4 - 7x^2 + 7$ is even function of x.

Q.6: If $\underset{x \to 2}{\text{Lt}} \frac{x^n - 2^n}{x - 2} = 80$, Find the value of n

Q.7: Find the value of $\lim\limits_{x \to 2} \frac{x^3 - 8}{x^2 - 3x + 2}$

Q.8: Find $\underset{x \to 0}{\text{Lt}} \frac{\sqrt{1 + x} - 1}{x}$ Q.9: $\lim\limits_{x \to 3} \frac{x - 3}{x^2 - 9}$

Q.10 $\lim\limits_{x \to 1} \frac{x^3 - 1}{x - 1}$

Q.11: Find $\underset{x \to \infty}{\text{Lt}} \frac{(2x - 3)(3x - 4)}{(4x - 5)(5x - 6)}$

Q.12: Find $\underset{x \to \infty}{\text{Lt}} \left(1 - \frac{2x}{3}\right)^{1/x}$

Q.13: Find $\underset{n \to \infty}{\text{Lt}} \left(1 + \frac{1}{n + 1}\right)^n$

Q.14: Find $\underset{x \to 0}{\text{Lt}} \frac{\sin x^\circ}{x}$

Q.15: Find the value of $\underset{x \to 0}{\text{Lt}} \left(1 + \frac{x}{3}\right)^{1/x}$

Answers

Q.1: $\frac{3 - 7x + 4x^2}{x^2}$ Q.2: $\sin 2\theta$ (Ans.) Q.3: $\sin 2A$ Q.4: $\frac{4}{5}$

Q.6: $n = 5$ Q.7: 12 Q.8: $\frac{1}{2}$ Q.9: 1/6 Q.10: 3

Q.11: $\frac{3}{10}$ Q.12: $e^{-2/3}$ Q.13: e Q.14: $\frac{\pi}{180}$ Q.15: $e^{1/3}$

Objective Type Questions

Q.1: Encircle the correct one of the given answers in each item.

1. A function $f(x) = x^2 + 2x + 3$ is
   - (a) Odd
   - (b) Even
   - (c) Implicit
   - [✓ (d) Explicit]

S.P

2. Which one is the periodic function:
   - (a) $x^2 + 1$
   - (b) $2x$
   - [✓ (c) Sin x]
   - (d) $x^3 + 1$

3. Given $f(x) = \frac{1}{x} - 1$, then $f(2) =$
   - (a) 1
   - (b) 2
   - [✓ (c) $-\frac{1}{2}$]
   - (d) 3

S.P

4. If $f(x) = 3^x - 1$ 1 then $f(3) =$
   - (a) 27
   - (b) 8
   - [✓ (c) 26]
   - (d) 16

5. $\lim_{x \to 0} (1 + x)^{1/x}$
   - (a) 0
   - (b) 1
   - [✓ (c) e]
   - (d) e^2

6. $\lim_{n \to 2} (x - 1) =$
   - [✓ (a) 1]
   - (b) 2
   - (c) 3
   - (d) 4

7. $\lim_{n \to 2} cx =$
   - [✓ (a) 2c]
   - (b) c
   - (c) xc
   - (d) x

8. $\lim_{x \to \pi/3} \cos x =$
   - [✓ (a) $\frac{\sqrt{3}}{2}$]
   - (b) $\frac{1}{2}$
   - (c) 0
   - (d) $\frac{1}{\sqrt{2}}$

9. $\lim_{\theta \to \pi/2} \frac{\sin\theta}{\theta} =$

(a) 1 (b) $\frac{\pi}{2}$

(c) $\frac{2}{\pi}$ (d) $\frac{1}{2}$

10. $\lim\limits_{\theta \to \pi/2} \frac{1}{\text{Cos}\theta} =$

(a) 0 (b) ∞

(c) 1 (d) $\frac{2}{\pi}$

Answers

Q.1:	(1)	d	(2)	c	(3)	c	(4)	c
	(5)	c	(6)	a	(7)	a	(8)	a
	(9)	c	(10)	b				

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Chapter 2

Derivatives

2.1 Introduction:

Derivatives are used widely in science, economics, medicine and computer science to calculate velocity and acceleration, to explain the behavior of machinery, to estimate the drop in water levels as water is pumped out of tank and to predict the consequences of making cross in measurements.

Finding derivatives by evaluating limits can be lengthy and difficult. In this chapter we develop techniques to make calculating derivatives easier.

Increments :

If a variable x changes from one fixed value x_1 to another x_2, the difference $x_2 - x_1$ is called an **increment** of x. In general an incement of x may be positive or negative , and it is denoted by the symbol δx, and read as "delta x". Similarly, δy denotes as incement of y

2.2 Definition of a Derivative:

Let $y = f(x)$ be a function of x , then

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$, if it exists.

is called the **differential coefficient or derivative** of y with respect to x and it is denoted by $\frac{dy}{dx}$ or y' or $f'(x)$ or $y^{(1)} = D_y$, where $D = \frac{d}{dx}$.

The steps for obtaining a derivative are as follows:

Consider the function $y = f(x)$.

1. Give increments to x and y to obtain $y + \delta y = f(x + \delta x)$.
2. Subtract to obtain $\delta y = f(x + \delta x) - f(x)$
3. Divide by δ x to obtain $\frac{\delta y}{\delta x} = \frac{f(x + \delta y) - f(x)}{\delta x}$
4. Take the Limit to obtain $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$

If the limit exists it is called the derivative or the differential coefficient of the function f(x) w.r.t. x , it is denoted by $\frac{dy}{dx}$ or $f'(x)$.

The process to find the derivative from the above four steps is called differentiation by ab-initio method or by definition or from first principles.

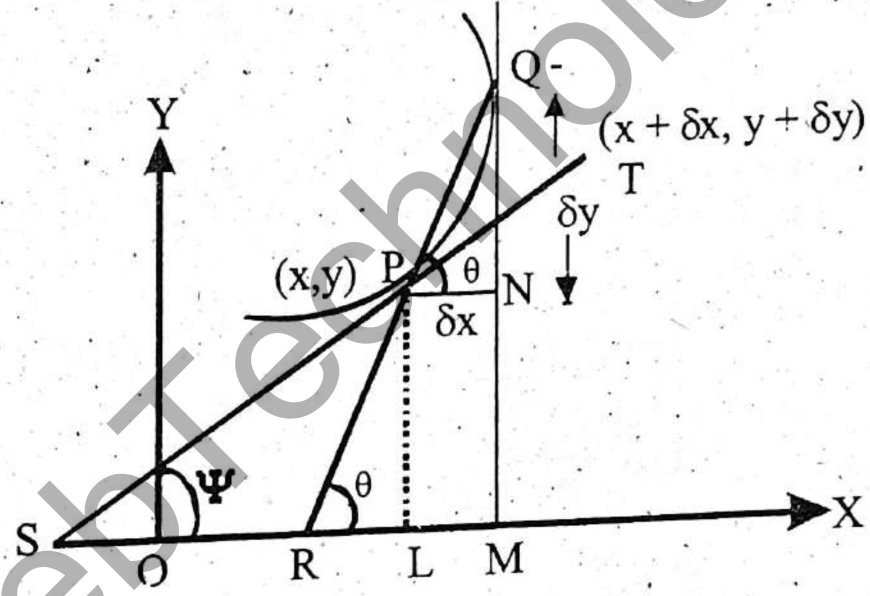
Secant Line:

Any line through two points on a curve is called a secant line.

2.3 Geometrical Interpretation of a derivative – Definition of a Derivative

If y = f(x), then the value of dy/dx at x = a is the slope of the Tangent of the curve represented by y =f(x).

Let P (x, y) and Q (x + δx, y + δy) be any two points on the curve y = f(x). Join QP and produce it to meet the x-axis at R.



PL, QM and PN have been drawn perpendiculars as shown in the figure. Clearly, PN = δx, NQ = δy.

If $\angle QPN = \theta$ then $\tan\theta = \frac{NQ}{PN} = \frac{\delta y}{\delta x}$. As $\delta x \to 0$, $Q \to P$ and the line QPR becomes tangent at P. Suppose the angle that the tangent at P makes with the x-axis at S is denoted by Ψ, then

$$\tan\Psi = \underset{Q\to P}{\text{Lt}} \tan\theta = \underset{\delta x\to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

i.e., slope (or gradient) of the tangent at P(x,y) = $\frac{dy}{dx}$, the value of the differential co-efficient at P(x, y). This way of stating the slope of a curve is called Leibnitz notation.

The slope of the curve at P = $\lim\limits_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = f'(x)$

This way of stating the slope of a curve is called Functional notation.

Examples1:

Differentiate by first principle (ab-initio) w.r.t. x

(i) $y = x$ (ii) $y = \frac{1}{x^3}$ (iii). $y = \sqrt{x}$

(iv). $y = \frac{1}{\sqrt{x+1}}$ (v). $y = x^2 + \frac{1}{x^2}$

Solution: (i)

$$y = x$$
$$y + \delta y = x + \delta x$$
$$y + \delta y - y = x + \delta x - x \quad \Rightarrow \delta y = \delta x$$
$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x}$$
$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} 1$$
$$\frac{dy}{dx} = 1$$

Solution: (ii) $y = \frac{1}{x^3}$

$$y = \frac{1}{x^3}$$
$$y + \delta y = \frac{1}{(x+\delta x)^3}$$
$$y + \delta y - y = \frac{1}{(x+\delta x)^3} - \frac{1}{x^3}$$

$$\delta y = \frac{x^3 - [x^3 + 3x^2(\delta x) + 3x\,(\delta x)^2 + (\delta x)^3]}{x^3(x+\delta x)^3}$$

$$= \frac{-3x^2\delta x - 3x(\delta x)^2 - (\delta x)^3}{x^3(x+\delta x)^3}$$

$$\frac{\delta y}{\delta x} = \frac{-3x^2 - 3x(\delta x) - (\delta x)^2}{x^3(x+\delta x)^3}$$

$$\lim_{x\to\infty}\frac{\delta y}{\delta x} = \lim_{x\to 0}\frac{-3x^2 - 3x(\delta x) - (\delta x)^2}{x^3(x+\delta x)^3}$$

$$\frac{dy}{dx} = \lim_{x\to 0}\frac{-3x^2}{x^6} = -\frac{3}{x^4}$$

Solution (iii) $y = \sqrt{x}$

$$y + \delta y = \sqrt{x+\delta x}$$

$$y + \delta y - y = \sqrt{x+\delta x} - \sqrt{x}$$

$$\delta y = \sqrt{x+\delta x} - \sqrt{x}$$

$$\frac{\delta y}{\delta x} = \frac{\sqrt{X+\delta X} - \sqrt{X}}{\delta X}$$

$$\frac{\delta y}{\delta x} = \frac{\sqrt{X+\delta X} - \sqrt{X}}{\delta X} \times \frac{\sqrt{X+\delta X} + \sqrt{X}}{\sqrt{X+\delta X} + \sqrt{X}} \quad \text{(By rationalization)}$$

$$= \frac{X + \delta X - X}{\delta X(\sqrt{X+\delta X} + \sqrt{X})}$$

$$\lim_{x\to 0}\frac{\delta y}{\delta x} = \lim_{x\to 0}\frac{1}{\sqrt{x+\delta x} + \sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Solution : (iv). $y = \dfrac{1}{\sqrt{x+1}} \Rightarrow y + \delta y = \dfrac{1}{\sqrt{x+\delta x+1}}$

$$y + \delta y - y = \frac{1}{\sqrt{x+\delta x+1}} - \frac{1}{\sqrt{x+1}}$$

$$\delta y = \frac{\sqrt{x+1} - \sqrt{x+\delta x+1}}{\sqrt{x+\delta x+1}\ \sqrt{x-1}} \text{ x } \frac{\sqrt{x+1} + \sqrt{x+\delta x+1}}{\sqrt{x+1} + \sqrt{x+\delta s+1}}$$

$2+\frac{1}{2}$ $4+1$ $\frac{5}{2}$ $\frac{1}{2}+1$ $1+\frac{2}{2} \Rightarrow \frac{3}{2}$

$$\delta y = \frac{x+1-x-\delta x-1}{\sqrt{x+\delta x+1}\ .\ \sqrt{x+1}\,(\sqrt{x+1}+\sqrt{x+\delta x+1})}$$

$$\lim_{x\to 0}\frac{\delta y}{\delta x} = \lim_{x\to 0}\frac{-1}{\sqrt{x+\delta x+1}\sqrt{x+1}\,[\sqrt{x+1}+\sqrt{x+\delta x+1}]}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{x+1}\sqrt{x+1}\,[\sqrt{x+1}+\sqrt{x+1}]}$$

$$\frac{dy}{dx} = \frac{-1}{(x+1)\,(2\sqrt{x+1})} = \frac{-1}{2(x+1)^{3/2}}$$

$2+\frac{1}{2}$ $4+1$

Solution: (v). $y = x^2 + \frac{1}{x^2}$

$$y+\delta y = (x+\delta x)^2 + \frac{1}{(x+\delta x)^2}$$

$$y+\delta y-y = (x+\delta x)^2 + \frac{1}{(x+\delta x)^2} - x^2 - \frac{1}{x^2}$$

$$\delta y = (x+\delta x)^2 + \frac{1}{(x+\delta x)^2} - x^2 - \frac{1}{x^2}$$

$$(x+\delta x)^2 - x^2 + \frac{1}{(x+\delta x)^2} - \frac{1}{x^2}$$

$$\delta y = x^2 + (\delta x)^2 + 2x\,(\delta x) - x^2 + \frac{x^2-(x+\delta x)^2}{(x+\delta x)^2\,x^2}$$

$$\delta y = x^2 + (\delta x)^2 + 2x\,(\delta x) + \frac{x^2-x^2-(\delta x)^2-2x(\delta x)}{(x+\delta x)^2\,x^2}$$

$$\delta y = (\delta x)^2 + 2x(\delta x) - \frac{\delta x\,(\delta x+\delta x)}{(x+\delta x)^2\,x^2}$$

$$\frac{\delta y}{\delta x} = \frac{\delta x[(\delta x+2x)]}{\delta x} - \frac{\delta x\,(\delta x+2x)}{\delta x\,(x+\delta x)^2\,x^2}$$

$$\lim_{x\to 0}\frac{\delta y}{\delta x} = \lim_{x\to 0}(\delta x+2x) - \lim_{x\to 0}\frac{\delta x+2x}{(x+\delta x)^2\,x^2}$$

$$\frac{dy}{dx} = 2x - \frac{2x}{x^2x^2} = 2x - \frac{2x}{x^4}$$

$$\frac{dy}{dx} = 2x - \frac{2}{x^3}$$

Exercise 2.1

$-\frac{1}{2}-1$

Q.1: Differentiate w.r.t x ab-initio.

i. x^3 ii. $\frac{1}{x^2}$ iii. $\frac{1}{\sqrt{x}}$

iv. $x^{\frac{2}{3}}$ v. $x^{\frac{1}{4}}$ vi. $x^{-\frac{4}{5}}$

Q.2: Find the derivative from first principle.

i. x^3-3x+4 ii. $\sqrt{x+9}$ iii $(x+4)^{\frac{2}{3}}$

Answers 2.1

Q.1: i. $3x^2$ ii. $\frac{-2}{x^3}$ iii. $-\frac{1}{2x^{3/2}}$

iv. $\frac{2}{3x^{1/3}}$ v. $\frac{1}{4x^{3/4}}$ vi. $-\frac{4}{5}x^{-9/5}$

Q.2 i. $3x^2-3$ ii. $\frac{1}{2\sqrt{x+9}}$ iii $\frac{2}{3}(x+4)^{-\frac{1}{3}}$

2.4 Fundamental Rules for Differentiation

Rule 1:

Derivative of x^n (power rule)

Let $y = x^n$

$$y+\delta y=(x+\delta x)^n$$

$$\delta y=(x+\delta x)^n-x^n \quad = x^n\left(1-\frac{\delta x}{x}\right)^n-x^n$$

$$=x^n\left[\left(1+\frac{\delta x}{x}\right)^n-1\right]$$

$$=x^n\left[1+n\frac{\delta x}{x}+\frac{n(n-1)}{2!}\left(\frac{\delta x}{x}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{\delta x}{x}\right)^3+\ldots\ldots-1\right]$$

$$\delta y = x^n\left[n\frac{\delta x}{x} + \frac{n(n-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{\delta x}{x}\right)^3 + \ldots\ldots\ldots\ldots\ldots\right]$$

$$\frac{\delta y}{\delta x} = \frac{x^n\,\delta x}{\delta x}\left[n\frac{1}{x} + \frac{n(n-1)\delta x}{2!\quad x^2} + \ldots\ldots\ldots\right]$$

$$\lim_{x\to 0}\frac{\delta y}{\delta x} = x^n \lim_{x\to 0}\left[n\frac{1}{x} + \frac{n(n-1)\delta x}{2!\quad x^2} + \ldots\ldots\ldots\right]$$

$$\frac{dy}{dx} = x^n\left[n\frac{1}{x} + 0 + 0\ldots\ldots\ldots\right]$$

$$\frac{dy}{dx} = nx^{n-1}$$

$$\therefore \quad \frac{dy}{dx} = (x^n) = n\,x^{n-1}$$

Note : if n is positive , we use Binomial theorem.

Examples 2:

i. $y = x^7$ ii. $y = x^{3/4}$ Find $\frac{dy}{dx}$

Solution:

i. $y = x^7$

$$\frac{dy}{dx} = 7\,x^{7-1} = 7x^6$$

ii. $y = x^{3/4}$

$$\frac{dy}{dx} = \frac{3}{4}x^{3/4-1} = \frac{3}{4}x^{-1/4} = \frac{3}{4\,x^{1/4}}$$

Rule 2:

The differential coefficient of any constant is zero.

Let $y = c$

$$y + \delta y = c$$

$$y + \delta y - y = c - c$$

$$\delta y = 0$$

$$\lim_{x\to 0}\frac{\delta y}{\delta x} = 0 \implies \frac{dy}{dx} = 0$$

Rule 3:

Product of constant and function.

Let $y = cv$

Where c is a constant and v is a function of x.

$$y + \delta y = c(v + \delta v)$$

$$\delta y = c(v + \delta v) - cv$$

$$\delta y = cv + c\delta v) - cv$$

$$\delta y = c\,\delta v$$

$$\frac{\delta y}{\delta x} = c\,\frac{\delta v}{\delta x}$$

$$\lim_{x\to 0}\frac{\delta y}{\delta x} = \lim_{x\to 0}\left(c.\frac{\delta v}{\delta x}\right)$$

$$\frac{dy}{dx} = c\,\frac{dv}{dx}$$

Example 3:

If $y = 3x^2$, find $\frac{dy}{dx}$

Solution:

$$y = 3x^2$$

$$\frac{dy}{dx} = \frac{d}{dx}(3x^2) = 3.\frac{d}{dx}(x^2) = 3.2x. = 6x$$

Rule 4 : Differential coefficient of a sum.

The differential coefficient of the sum of a set of functions of x is the sum of the differential coefficients of the several functions.

Let $y = u + v + w + \ldots\ldots\ldots\ldots$

Which are all finite in number and function of x, therefore, y is also function of x.

Let the increments of u, v, w..........and y be respectively.

$\delta u, \delta v, \delta w, \ldots\ldots\ldots\ldots, \delta y$

$$\therefore\quad y + \delta y = (u + \delta u) + (v + \delta v) + (w + \delta w)\ldots\ldots\ldots\ldots$$

$$\delta y = \delta u + \delta v + \delta w + \ldots\ldots\ldots\ldots$$

$$\frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x} + \frac{\delta w}{\delta x} + \ldots\ldots\ldots\ldots$$

$$\therefore \quad \lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{\delta u}{\delta x}+\lim_{\delta x\to 0}\frac{\delta v}{\delta x}+\lim_{\delta x\to 0}\frac{\delta w}{\delta x}$$

Hence $$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}+\ldots\ldots\ldots\ldots$$

Example 4:

If $y=x^3+x^2+2x+3$ find $\frac{dy}{dx}$

Solution:

$$y=x^3+x^2+2x+3$$

$$\frac{dy}{dn}=\frac{d}{dn}(x^3+x^2+2x+3)$$

$$=\frac{dy}{dn}(x^3)+\frac{d}{dn}(x^2)+2\frac{d}{dn}(x)+\frac{d}{dn}(3)$$

$$=3x^2+2x+2$$

Rule 5 (Chain rule):

Differentiation of a function of function or composite functions:

When y is a function of u and u is further a function of x. then y is also function of x and called function of function or composite function. The derivative of y w.r.t. x is product of derivative of y w.r.t. u and the derivative of u w.r.t. x. (This is sometimes called the chain rule of differentiation)

Since $y=f(u)$ and $u=g(x)$, then $\frac{dy}{dx}=\frac{dy}{du}\,\frac{du}{dx}$

Proof:

Let δx, δu and δy be the increments of x, u and y then

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta u}\,\frac{\delta u}{\delta x}$$

In the limiting case as δx →0 also δu →0

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta u\to 0}\frac{\delta y}{\delta u}\cdot\lim_{\delta x\to 0}\frac{\delta u}{\delta x}$$

$$\frac{dy}{dx}=\frac{dy}{du}\,\frac{du}{dx}$$

Example 5:

If $y = (3x^2 + 2x + 9)^7$ Find $\frac{dy}{dx}$

Solution:

Let $u = 3x^2 + 2x + 9$

Then $y = u^7$ 7

$$\frac{dy}{du} = 7u^6$$

also $u = 3x^2 + 2x + 9$

$$\frac{du}{dx} = 6x + 2$$

Using Chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 7u^6 (6x + 2)$$

$$= 7(3x^2 + 2x + 9)^6 (6x + 2)$$

Rule 6:

Generalization of Power rule

Let $y = [f(x)]^n$

Put $u = f(x)$ then $y = u^n$

And $\frac{du}{dx} = f'(x)$ $\frac{dy}{du} = nu^{n-1}$

By chain rule $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$

$$= nu^{n-1} f'(x)$$

$$\frac{dy}{dx} = n[f(x)]^{n-1} f'(x)$$

Example 6:

Find the derivative of $(ax + b)^n$

Solution:

Let $y = (ax + b)^n$

$$y + \delta y = [a(x + \delta x) + b]^n$$

$$= [a(x + \delta x) + b]^n = [\ (ax + b) + a\ \delta x]^n$$

$$= (ax + b)^n \left(1 + \frac{a\delta x}{ax + b}\right)^n$$

$$y + \delta y - y = (ax + b)^n \left(1 + \frac{a\delta x}{ax + b}\right)^n - (ax + b)^n$$

$$\delta y = (ax + b)^n \left[\left(1 + \frac{a\delta x}{ax + b}\right)^n - 1\right]$$

$$= (ax + b)^n \left[1 + n\frac{a\delta x}{ax + b} + \frac{n(n-1)}{2!}\left(\frac{a\delta x}{ax + b}\right)^2 + \ldots\ldots -1\right]$$

$$\delta y = (ax + b)^n \left[n\frac{a\delta x}{ax + b} + \frac{n(n-1)}{2!}\left(\frac{a\delta x}{ax + b}\right)^2 + \ldots\ldots\right]$$

Divide both sides by δx and taking limit.

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} (ax + b)^n \left[n\left(\frac{a}{ax + b}\right) + \frac{n(n-1)}{2!}\frac{a^2\delta x}{(ax + b)^2} + \ldots\ldots\right]$$

$$\frac{dy}{dx} = (ax + b)^n \left[\frac{na}{ax + b} + 0 + \ldots\ldots\right]$$

$$\frac{dy}{dx} = n(ax + b)^{n-1} a$$

Note : if n is positive , we use binomial theorem.

OR

$y = (ax + b)^n$

Apply the generalization of power rule.

$$\frac{dy}{dx} = (ax + b)^{n-1} \cdot \frac{d}{dx}(ax + b)$$

$$\frac{dy}{dx} = (ax+b)^{n-1}(a+0)$$

$$= a(ax+b)^{n-1}$$

Example 7:

$$y = \sqrt{x^2+1} \quad \text{find} \quad \frac{dy}{dx}$$

Solution:

$$\therefore \quad y = (x^2+1)^{1/2}$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2+1)^{1/2}$$

$$= \frac{1}{2}(x^2+1)^{-1/2}\frac{d}{dx}(x^2+1)$$

$$= \frac{1}{2}(x^2+1)^{1/2}\,2x$$

$$= \frac{x}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}(x^2+1)^{1/2}$$

$$\frac{x}{\sqrt{x^2+1}}$$

Rule 7:

The Differential Coefficient of the Product of Two Functions:

The product of two derivable functions is itself derivable and its derivative equal to (first function. derivative of second function) + (Second function. derivative of first function)

Let $y = uv$

Where u, v are two derivable function of x.

Let δu, δv, δy be the increments in u, v and y respectively corresponding to the increment δx in x. We have

$$y + \delta y = (u+\delta u)(v+\delta v)$$

$$= uv + u\delta v + v\delta u + \delta u\ \delta v$$

$$y + \delta y - y = uv + u\delta v + u\delta u + \delta u.\ \delta v - uv$$

$$\delta y = u.\delta v + v.\delta u + \delta u.\ \delta v$$

$$\frac{\delta y}{\delta x} = u.\frac{\delta v}{\delta x} + v.\frac{\delta u}{\delta x} + \delta u.\frac{\delta v}{\delta x}$$

When $\delta x \to 0$. Then also $\delta u \to 0$

$$\lim_{\delta \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left[u \frac{\delta v}{\delta x} + v. \frac{\delta u}{\delta x} + \delta u. \frac{\delta v}{\delta x} \right]$$

$$\frac{dy}{dx} = u. \lim_{\delta x \to 0} \frac{\delta v}{\delta x} + v \lim_{x \to 0} \frac{\delta u}{\delta x} + \lim_{\delta x \to 0} \delta u. \frac{\delta v}{\delta x}$$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v. \frac{du}{dx}$$

$$\therefore \quad \frac{dy}{dx}(uv) = u. \frac{dv}{dx} + v. \frac{du}{dx}$$

Note: Extension rule of product: The result may be extended for the product of more than two functions.

Let $y = uvw$

Then $\frac{dy}{dx} = uv \frac{dw}{dx} + uw. \frac{dv}{dx} + vw. \frac{du}{dx}$

Example 8:

Find the derivatives of $y = (x^2 + 2)(x^3 + 3)$

Solution:

Here $u = x^2 + 2$ and $v = x^3 + 3.$ Then $y = uv$

$\frac{du}{dx} = 2x$ $\qquad$ $\frac{dv}{dx} = 3x^2$

$$\frac{dy}{dx} = u. \frac{dv}{dx} + v. \frac{du}{dx}$$
$$= (x^2 + 2)(3x^2) + (x^3 + 3)(2x)$$
$$= 3x^4 + 6x^2 + 2x^4 + 6x$$
$$= 5x^4 + 6x^2 + 6x$$

Rule 9:

The differential Coefficient of a Quotient of Two Function:

Let $y = \frac{u}{v}$

Where u, v are two derivable function of x also v is not zero for all values under consideration.

Then $y + \delta y = \frac{u + \delta u}{v + \delta v}$

$$y + \delta y - y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v}$$

$$\Rightarrow \qquad \delta y = \frac{v.\delta u - u.\ \delta u}{v(v + \delta v)}$$

$$\frac{\delta y}{\delta x} = \left[\frac{v\dfrac{\delta u}{\delta x} - u.\dfrac{\delta v}{\delta x}}{v(v - \delta v)}\right]$$

When $\delta x \rightarrow 0$ also $\delta v \rightarrow 0$:

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left[\frac{v\dfrac{\delta u}{\delta x} - u.\dfrac{\delta v}{\delta x}}{v^2}\right]$$

$$\frac{d}{dx} = \frac{v\dfrac{du}{dx} - u.\dfrac{dv}{dx}}{v^2}$$

$$\therefore \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u.\dfrac{dv}{dx}}{v^2}$$

Example 9:

$\Rightarrow$ If $y = \frac{x}{1 + x}$ find $\frac{dy}{dx}$

Solution:

$$\frac{dy}{dx} = \frac{(1 + x)\dfrac{d}{dx}(x) - x.\dfrac{d}{dx}(1 + x)}{(1 + x)^2}$$

$$= \frac{(1 + x)\,1 - x.1}{(1 + x)^2}$$

$$= \frac{1 + \cancel{x} - \cancel{x}}{(1 + x)^2}$$

$$= \frac{1}{(1 + x)^2}$$

Illustrative Examples:

1. Find the derivatives of the following:

i. $x^3 + 4x + 9$ ii. $x^{3/2} + x^{1/2} + 1$

$\frac{3}{2} - 1 \qquad \frac{1}{2} - 1$

$= \frac{3-2}{2} = \frac{1}{2} \qquad = \frac{1-2}{2} = -\frac{1}{2}$

$= \frac{3}{2} x^{\frac{3}{2}-1} + \frac{1}{2} x^{\frac{1}{2}-1}$

$= \frac{3}{2} x^{\frac{1}{2}} + \frac{1}{2} x^{-\frac{1}{2}}$

$= \frac{3}{2} x^{1/2} + \frac{1}{2x^{1/2}}$

Solution:

i. Let $y = x^3 + 4x + 9$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 + 4x + 9)$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) + 4\frac{d}{dx}(x) + \frac{d}{dx}(9)$$

$$\frac{dy}{dx} = 3x^3 + 4 + 0$$

$$\frac{dy}{dx} = 3x^2 + 4$$

ii. Let $y = x^{3/2} + x^{1/2} + 1$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{3/2} + x^{1/2} + 1\right)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{3/2}\right) + \frac{d}{dx}\left(x^{1/2}\right) + \frac{d}{dx}(1)$$

$$= \frac{3}{2}x^{3/2-1} + \frac{1}{2}x^{1/2-1} + 0$$

$$= \frac{3}{2}x^{1/2} + \frac{1}{2x^{1/2}}$$

2. i. Find $\frac{dy}{dx}$ at $x = 2$ When $y = \sqrt{u^2 - 2}$ and $u = 4\sqrt{x}$

ii. if $y = \sqrt{ax^2 + 2bx + c}$ Find $\frac{dy}{dx}$

Solution:

i. $y = \sqrt{u^2 - 2}$ $u = 4\sqrt{x}$

$\frac{1}{2}-1$

$=\frac{1-2}{2}$

$=-\frac{1}{2}$

$$\frac{dy}{du}=\frac{1}{2}\,(u^2-2)^{-1/2}(2u)\ ,\qquad \frac{du}{dx}=\overset{2}{\cancel{4}}\,\frac{1}{\cancel{2}}x^{-1/2}$$

$$\frac{dy}{du}=\frac{u}{\sqrt{u^2-2}}\ ,\qquad \frac{du}{dx}=\frac{2}{\sqrt{x}}$$

Using chain rule

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$\frac{d}{dx}(u^2-2)^{1/2}$

$$=\frac{u}{\sqrt{u^2-2}}\ \frac{2}{\sqrt{x}}$$

$\frac{1}{2}(u^2-2)^{-1/2}\cdot\frac{d}{d}$

$$=\frac{4\sqrt{x}}{\sqrt{16x-2}}\ \frac{2}{\sqrt{x}}$$

$$=\frac{8}{\sqrt{16x-2}}$$

at $x=2$

$\frac{1}{2(u^2-2)^{1/2}}$

$$\therefore\left(\frac{dy}{dx}\right)_2=\frac{8}{\sqrt{32-2}}=\frac{8}{\sqrt{30}}$$

ii. Let $u=ax^2+2bx+c$

$$y=\sqrt{u}\ ,\qquad u=ax^2+2bx+c$$

$\frac{1}{2\sqrt{(4\sqrt{x})^2-2}}$

$$\Rightarrow\frac{dy}{du}=\frac{1}{2}(u)^{-1/2}\ ,\qquad \frac{du}{dx}=2ax+2b$$

$$=\frac{1}{2\sqrt{u}}$$

$=\frac{1}{2\sqrt{16x-2}}$

$\frac{1}{2}(ax^2+2bx+c)^{-1/2}\cdot(2ax+2b)$

Using chain rule

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$\frac{1}{2}\ \frac{2ax+2b}{\sqrt{ax^2+2bx+c}}$

$$=\frac{1}{2\sqrt{u}}\times(2ax+2b)$$

$$=\frac{ax+b}{\sqrt{ax^2+2bx+c}}$$

$\frac{2(ax+b)}{2\sqrt{ax^2+2bx+c}}=\frac{ax+b}{\sqrt{ax^2+2bx+c}}$

Summary

1. (i) $\frac{d}{dx}(x^n) = nx^{n-1}$, for all constant values of n (ii) $\frac{d}{dx}(x) = 1$.

2. $\frac{d}{dx}(c) = 0$, where c is any constant.

3. $\frac{d}{dx}(u+v-w+\ldots) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx} + \ldots$

4. (i) $\frac{d}{dx}(cu) = c\frac{du}{dx}$ (ii) $\frac{d}{dx}(\frac{u}{c}) = \frac{1}{c}\frac{du}{dx}$

5. $\frac{d}{dx}(ax+b)^n = na\,(ax+b)^{n-1}$, for all constant values of n.

6. $\frac{d}{dx}(v)^n = nv^{n-1}\frac{dv}{dx}$, where v is a function of x.

7. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

8. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

9. $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$, y being a function of u, u being function of x.

10. $\frac{dy}{dx} = \frac{1}{dx/dy}$, y being a function of x.

In above results u, v, w, y are functions of x whereas a, b, c and n are constants.

Exercise 2.2

Q.1 Differentiate the following w.r.t.x.

i. $7x^7$ ii. $-5 + 3x - \frac{3}{2}x^2 - 7x^3$ iii. $2x^3 + 4x^2 - 5x + 8$

iv. $3x^2 + x^{-1/2}$ v $\frac{6}{x} + \frac{4}{x^2} - \frac{3}{x^3}$ vi. $\frac{1}{5}x^{5/2} + \frac{1}{3}x^{3/2}$

vii $x^2 - x^{-2}$ viii $\sqrt{x} + \frac{1}{\sqrt{x}}$ ix $\sqrt{x}\,(x^3 - 1)$

Q.2 **Differentiate w.r.t.x**

i. $\sqrt{x^2+1}$ ii. $(x^2+3x+9)^{3/2}$ iii. $(ax^p+bx^q)^{p/q}$

iv. $\sqrt[3]{x^2+9x+8}$ v. $\frac{1}{\sqrt{x+b}}$ vi. $\frac{1}{\sqrt{a^2-x^2}}$

vii. $(x+x^{-1})^2$ viii. $(1+x+x^2)^3$ ix $(1-x^2)^{1/2}$

x $(2x^2+4x-5)^6$

Q.3 **Find the derivative w.r.t.x**

i. $(4x^3+7)(3x^2+2)$ ii. $(ax^2+b)(cx^2+d)$ iii. $x\sqrt{x+1}$

iv. $(x-1)^3(x+2)^4$ v. $(x+1)^2(x^2+1)^{-3}$ vi. $(a+x)\sqrt{a-x}$

vii $(3-x^2)(x^3-x+1)$ viii $(x-1)(x^2+x+1)$

ix $(x+3)(2x+3)(x^2+1)$

Q.4. **Differentiate w.r.t.x.**

i. $\frac{x}{x^2+1}$ ii. $\frac{x+1}{x^2+2x+2}$ iii. $\frac{x^2}{1+x^2}$

iv. $\sqrt{\frac{1+x}{1-x}}$ v. $\sqrt{\frac{a+x}{a-x}}$ vi. $x\sqrt{\frac{a+x}{a-x}}$

vii. $\frac{\sqrt{x}}{\sqrt{x+1}}$ viii. $\frac{x}{(a^2+x^2)^{3/2}}$ ix. $\left(\frac{x+1}{x-1}\right)^2$

x. $\frac{(x^2+1)^3}{(x^2+2)^2}$

Q.5 **Find dy/dx at the given point if:**

(i) $y=x^{2/3}$ at $x=8$ (ii) $y=8x-3x^2$ at $x=2$

(iii) $y=x^4-x^2+2$ at $x=-1$ (iv) $y=x+2x^{-1}$ at $x=2$

Answers 2.2

Q.1

(i) $49x^6$ (ii) $3-3x-21x^2$ (iii) $6x^2+8x-5$

(iv) $6x-\frac{1}{2}x^{-3/2}$ (v) $-\frac{6}{x^2}-\frac{8}{x^3}+\frac{9}{x^4}$ (vi) $\frac{1}{2}x^{3/2}+\frac{1}{2}x^{1/2}$

vii $2x+2/x^3$ (viii) $\frac{x-1}{2x^{3/2}}$ (ix) $\frac{7}{2}x^{5/2}-\frac{1}{2\sqrt{x}}$

Q.2

(i) $\frac{x}{\sqrt{x^2+1}}$ (ii) $\frac{3}{2}(x^2+3x+9)^{\frac{1}{2}}$ ~~$(3x+9)$~~ (2x+3)

(iii) $\frac{p}{q}(ax^p + bx^q)^{p/q-1}(ap.x^{p-1} + bqx^{q-1})$

(iv) $\frac{1}{3}(x^2+9x+8)^{-2/3}(2x+9)$

(v) $\frac{-1}{2(x+b)^{3/2}}$ (vi) $\frac{x}{(a^2-x^2)^{3/2}}$

(vii) $\frac{2(x^4-1)}{x^3}$ (viii) $3(1+2x)(1+x+x^2)^2$

(ix) $\frac{-x}{\sqrt{1-x^2}}$ (x) $24(x+1)(2x^2+4-5)^5$

Q.3

(i) $6x(10x^3+4x+7)$ (ii) $2x(2acx^2+bc+ad)$ (iii) $\frac{3x+2}{2\sqrt{x+1}}$

(iv) $(x-1)^2(x+2)^3(7x+2)$ (v) $\frac{-2(2x^3+5x^2+2x-1)}{(x^2+1)^4}$

(vi) $\frac{a-3x}{2\sqrt{a-x}}$ (vii) $-5x^4+12x^2-2x-3$ (viii) $3x^2$

(ix) $8x^3+27x^2+22x+9$

Q.4

(i) $\frac{1-x^2}{(1+x^2)^2}$ (ii) $\frac{-x(x+2)}{(x^2+2x+2)^2}$ (iii) $\frac{2x}{(1+x^2)^2}$

(iv) $\frac{1}{\sqrt{1+x}\,(1-x)^{3/2}}$ (v) $\frac{a}{(a-x)\sqrt{a^2-x^2}}$

(vi) $\frac{-x^2+ax+a^2}{(a+x)^{1/2}(a-x)^{3/2}}$ (vii) $\frac{1}{2\sqrt{x}(x+1)^{3/2}}$ (viii) $\frac{a^2-2x^2}{(a^2+x^2)^{5/2}}$

(ix) $\frac{-4(x+1)}{(x-1)^3}$ (x) $\frac{2x(x^2+1)^3(x^2+4)}{(x^2+2)^3}$

Q.5: (i) 1/3 (ii) - 4 (iii) - 2 (iv) 1/2

2.5 Differentiation of Implicit Function

We have been concerned with the case in which y is expressed explicitly i.e. directly in terms of x. However, there are many cases in which y is not expressed directly in terms of x, but its functionality is implified by an algebraic relation connecting x and y.

2.5.1 Definition:

A function in which two or more variables are related to each other but are not independent to each other is called an implicit function.

An implicit function in x and y is expressed as f(x, y) = 0. For example:

(i) $x^2 - y^2 + xy + 3 = 0$ (ii) $x^3 + x^2y + y^3 = 7$

In such implicit relation we proceed as follows

Example 1:

If $x^3 + y^3 = 3axy$ find $\frac{dy}{dx}$

Solution:

Diff. both sides w.r.t.x.

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(y.1 + x.\frac{dy}{dx}\right)$$

$$3(x^2 - ay) + 3(y^2 - ax)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{(x^2 - ay)}{y^2 - ax}$$

$3x^2+3y^2\frac{dy}{dx}-3ay-3ax\frac{dy}{dx}=0$

$3(x^2-ay)+3(y^2-ax)\frac{dy}{dx}$

Example 2:

Find $\frac{dy}{dx}$ from the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Solution: Diff. w.r.t.x.

$$2ax + 2h\left(x.\frac{dy}{dx} + y\right) + 2by.\frac{dy}{dx} + 2g + 2f.\frac{dy}{dx} = 0$$

$$(hx + by + f)\frac{dy}{dx} = -(ax + hy + g)$$

$$\frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}$$

Where, $hx + by + f \neq 0$

2.6 Differentiation of Parametric Function:

Some time we deal with the functions in which dependent variable y is not given in terms of independent variable x rather both of them are functions of an other variable" t" called a parameter.

Its general form is $x = f(t)$ and $y = g(t)$

In order to differentiate parametric equations, we find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ from the given equations, then using chain rule find $\frac{dy}{dx}$.

i.e. $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

Example 1:

Find $\frac{dy}{dx}$ when $x = at^2$, $y = 2at$

Solution:

Diff. both equations w.r.t.t.

As, $x = at^2 \implies \frac{dx}{dt} = 2\,at$

and $y = 2\,at \implies \frac{dy}{dt} = 2a$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2a \times \frac{1}{2at} = \frac{1}{t}$$

Example 2:

If $x = 2\theta + 1,$ $y = (2\theta + 1)^2$ Find $\frac{dy}{dx}$

Solution:

Diff. both equation w.r.t. θ

$x = 2\theta + 1$, $y = (2\theta + 1)^2$

$\frac{dy}{d\theta} = 2$, $\frac{dy}{d\theta} = 2(2\theta + 1).2$

$= 4(2\theta + 1)$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= 4(2\theta + 1) \times \frac{1}{2}$$

$$= 2\,(2\theta + 1)$$

Exercise 2.3

Q.1: Find $\frac{dy}{dx}$ of the following:

i. $xy + y^2 = 2$
ii. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
iii. $x^{2/3} + y^{2/3} = a^{2/3}$
iv. $x\sqrt{1+y} + y\sqrt{1+x} = a$
v. $x^3 - 4xy^2 - 2y^2 - 5y^3 = 0$
vi $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{a}}$
vii $x^2 - 3y^2 + 7 = 0$
viii $x^3 + y^3 + 4 = 0$
ix $x^2 - 2xy + 2y^2 = 5$
x $x^3 + 6xy + 5y^3 = 3$

Q.2: Find $\frac{dy}{dx}$ of the following:

i. $y^2 = 4ax$
ii. $x^3 + y^3 = xy$
iii. $x^5 + y^5 = 5a^2x^2y^2$
iv. $y^2 = \frac{x-1}{x+1}$

Q.3: Find $\frac{dy}{dx}$

i. $x = t + 2, \quad y = 2t^2 + 2$
ii. $x = \theta^2 - \theta - 1, \quad y = 2\theta^2 + \theta + 1$
iii. $x = a\theta^3, \quad y = b\left(\theta - \frac{1}{\theta}\right)$
iv. $x = \frac{a(1-t^2)}{1+t^2}, \quad y = \frac{2bt}{1+t^2}$
v. $x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}$
vi. $x = a\left(\frac{t^2}{2} - t\right), \quad y = b\left(\frac{t^3}{3} - \frac{t^2}{2}\right)$
vii. $x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}$

Q.4: i. Show that if $x = a\theta^2, \quad y = 2a\theta$ then $y\frac{dy}{dx} - 2a = 0$

ii. if $y = x^4 + 2x^2$, prove that $\frac{dy}{dx} = 4x\sqrt{y+1}$

iii If $x = \frac{1-t^2}{1+t^2}, \quad y = \frac{t^2}{1+t^2}$ then prove that $y\frac{dy}{dx} + x = 0$

Answers 2.3

Q.1 **(i)** $\frac{-y}{x+2y}$ **(ii)** $-\frac{b^2x}{a^2y}$ **(iii)** $-\frac{y^{1/3}}{x^{1/3}}$

(iv) $\frac{-\sqrt{1+y}\left[y+2\sqrt{(1+x)(1+y)}\right]}{\sqrt{1+x}\left[x+2\sqrt{(1+x)(1+y)}\right]}$ (v) $\frac{3x^2-4y^2}{8xy+4y+15y^2}$

(vi) $-\left(\frac{y}{x}\right)^{3/2}$ (vii) $x/3y$ (viii) $-x^2/y^2$

(ix) $\frac{x-y}{x-2y}$ (x) $-\frac{x^2+2y}{2x+5y^2}$

Q.2 (i) $\frac{2a}{y}$ (ii) $\frac{3x^2-y}{x-3y^2}$ (iii) $\frac{x(2a^2y^2-x^3)}{y(y^3-2a^2x^2)}$

(iv) $\frac{1}{(x+1)^{3/2}\sqrt{x-1}}$

Q.3 **(i)** $4t$ **(ii)** $\frac{4\theta+1}{2\theta-1}$ **(iii)** $\frac{b}{3a}(\theta^{-2}+\theta^{-4})$

(iv) $\frac{b(t^2-1)}{2at}$ (v) $\frac{t(2-t^3)}{1-2t^3}$ (vi) $\frac{b}{a}t$

(vii) $\frac{t^2-1}{2t}$

2.7 Differentiation of Function w.r.t. Another Function

Sometime we differentiate given function w.r.t. another function instead of independent variable.

In this case we proceed as follows:

Example 1:

Differentiate x^3+8 w.r.t. x^2+4

Solution:

Let $y = x^3 + 8$ $u = x^2 + 4$

$$\frac{dy}{dx} = 3x^2 \qquad \frac{du}{dx} = 2x$$

Using chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$$

$$\frac{dy}{du} = 3x^2 \cdot \frac{1}{2x} = \frac{3x}{2}$$

Example 2:

Differentiate $\frac{x^2}{1+x^2}$ w.r.t. x^4

Solution:

Let $y = \frac{x^2}{1+x^2}$ and $u = x^4$

$$\frac{dy}{dx} = \frac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2}$$

$$= \frac{2x}{(1+x^2)^2}$$

$$\frac{du}{dx} = 4x^3$$

By chain Rule $\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dy}{du}$

$$\frac{dy}{du} = \frac{2x}{(1+x^2)^2} \times \frac{1}{4x^3}$$

$$\frac{dy}{du} = \frac{1}{2x^2(1+x^2)^2}$$

2.8 Differentiation by Rationalization

In order to differentiate irrational algebraic expression, it is useful to rationalize the given expression before differentiation on.

Example 3:

Differentiate w.r.t. x

$$y = \frac{\sqrt{a+x} - \sqrt{a-x}}{\sqrt{a+x} + \sqrt{a-x}}$$

Solution:

$$y=\frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}\times\frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$$ (By rationalization)

$$=\frac{a+x+a-x-2\sqrt{(a+x)(a-x)}}{a+x-a+x}$$

$$y=\frac{a-\sqrt{a^2-x^2}}{x}$$

$$\therefore\quad \frac{dy}{du}=\frac{x\frac{d}{dx}(a-\sqrt{a^2-x^2})-(a-\sqrt{a^2-x^2})\frac{d}{dx}(x)}{x^2}$$

$$=\frac{x\left\{-\frac{1}{2}(a^2-x^2)^{-1/2}(-2x)\right\}-(a-\sqrt{a^2-x^2})(1)}{x^2}$$

$$=\frac{\frac{x^2}{\sqrt{a^2-x^2}}+(\sqrt{a^2-x^2}-a)}{x^2}$$

$$=\frac{x^2+a^2-x^2-a\sqrt{a^2-x^2}}{x^2\sqrt{a^2-x^2}}$$

$$\frac{dy}{du}=\frac{a(a-\sqrt{a^2-x^2})}{x^2\sqrt{a^2-x^2}}$$

Example 4:

Differentiate $y=\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$

Solution:

$$y=\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\times\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$$

$$y=\frac{(\sqrt{1+x^2}+\sqrt{1-x^2})^2}{1+x^2-1+x^2}$$

$$= \frac{1+x^2+1-x^2+2\sqrt{(1+x^2)(1-x^2)}}{2x^2} = \frac{2+2\sqrt{1-x^4}}{2x^2} = \frac{1+\sqrt{1-x^4}}{x^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{x^2\frac{d}{dx}\left(1+\sqrt{1-x^4}\right)-\left(1+\sqrt{1-x^4}\right)\frac{d}{dx}(x^2)}{(x^2)^2}$$

$\frac{1}{2}-1$

$$= \frac{x^2\left\{\frac{-4x^3}{2\sqrt{1-x^4}}\right\}-2x\left(\frac{1+\sqrt{1-x^4}}{1}\right)}{x^4}$$

$\frac{1-2}{2}$

$$= \frac{-2x^5-2x\sqrt{1-x^4}-2x(1-x^4)}{x^4\sqrt{1-x^4}}$$

$-\frac{1}{2}$

$$= \frac{-2x^5-2x\sqrt{1-x^4}-2x+2x^5}{(x^2)^2}$$

$2x^2 - 2(x^4-1)$

$\sqrt{x^4-1}$

$$\frac{dy}{dx} = \frac{-2x\left(1+\sqrt{1-x^4}\right)}{x^4\sqrt{1-x^4}}$$

$x^2-\sqrt{x^4-1}$

$= 2x - \frac{1}{2}(x^4-1)^{-1/2}\frac{d}{dx}(x^4$

$2x - \frac{4x^3}{2\sqrt{x^4-1}} \Rightarrow 2x - \frac{x^3}{}$

Exercise 2.4

Q.1. Differentiate the following:

i. $\frac{x^2}{x^2-1}$ w.r.t. x^2 ii. $\frac{x^2+1}{x^2-1}$ w.r.t. $\frac{x-1}{x+1}$

iii. $2x^2+x+1$ w.r.t. x^2-x-1 iv. $x^2-\frac{1}{x^2}$ w.r.t. x^4

v. $\frac{ax+b}{cx+d}$ w.r.t. $\frac{ax^2+b}{cx^2+d}$ vi. $\frac{ax+b}{cx+d}$ w.r.t. $\frac{px+q}{rx+s}$

vii. $\frac{x^2+a^2}{x^2-a^2}$ w.r.t. $\frac{x-a}{x+a}$

Q.2: Find the derivative of the following:

$$\frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}} \Rightarrow \frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}} \times \frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}$$

$$\frac{\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)^2}{\left(\sqrt{x^2+1}\right)^2-\left(\sqrt{x^2-1}\right)^2} \Rightarrow \frac{x^2+\cancel{1}+x^2-\cancel{1}-2\sqrt{(x^2+1)(x^2-1)}}{\cancel{x^2}+1-\cancel{x^2}+1}$$

Answers 2.4

Q.1. i. $\frac{-1}{(x^2-1)^2}$ ii. $\frac{-2x}{(x-1)^2}$ iii. $\frac{4x+1}{2x-1}$

iv. $\frac{x^4+1}{2x^6}$ v. $\frac{(cx^2+d)^2}{2x(cx+d)^2}$

vi. $\frac{ad-bc}{ps-rq}\times\left(\frac{rx+s}{cx+d}\right)^2$ vii. $\frac{-2ax}{(x-a)^2}$

Q.2 $2x-\frac{2x^3}{\sqrt{x^4-1}}$

Short Questions

Write the short answers of the following:

Q.1: If $y = \sqrt{x} - \frac{1}{\sqrt{x}}$, Then show that $2x\frac{dy}{dx} + y = 2\sqrt{x}$

Q.2: Find $\frac{d}{dx}\left(\frac{1}{(ax+b)^m}\right)$

Q.3: If $y = x - \sqrt{x^2 + 1}$ Then show that $(y - x)\frac{dy}{dx} = y$

Q.4: if $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots$, Then show that $\frac{dy}{dx} = y$

Q.5: Find $\frac{dy}{dx}$ if $\sqrt{x} + \sqrt{y} = 5$

Q.6: Find $\frac{dy}{dx}$ if $y = x^3 + x^2 + 2x + 3$

Q.7: If $y = (3x^2 + 2x + 9)^7$, Find $\frac{dy}{dx}$

Q.8: If $y = 5x^3 - 7x^2 + 9 - \frac{8}{x} + \frac{7}{x^4}$, find $\frac{dy}{dx}$

Q.9: If $y = (a^2 + x^2)(b^3 + 3x^3)$, then Find $\frac{dy}{dx}$

Q.10: If $y = \frac{1}{(x-3)(x+2)}$, find $\frac{dy}{dx}$

Q.11: Diff. $\frac{x}{x^2+1}$ w.r.t. x

Q.12: :If $y = \frac{1+x}{1-x}$, find $\frac{dy}{dx}$

Q.13: If $y = \frac{x^2+1}{x-1}$ find $\frac{dy}{dx}$ at $x = 2$

Q.14: If $y = u^n$ and $u = (3x^3 - 7x^2 + x + 1)$, find $\frac{dy}{dx}$

Q.15: If $y = \sqrt{1 + x^2}$, show that $y\frac{dy}{dx} = x$

Q.16: If $x = \frac{2y-1}{y+3}$, Find $\frac{dy}{dx}$

Q.17: Find $\frac{dy}{dx}$ if $x^{2/3} + y^{2/3} = a^{2/3}$

Q.18: If $ax^2 + by^2 + 2h\,xy = 0$, find $\frac{dy}{dx}$

Q.19: Diff. $\frac{x^3}{1+x^3}$ w.r.t. x^3

Q.20: Differentiate $\frac{x^2}{1+x^2}$ W.r.t. x^2

Q.21: Find $\frac{dy}{dx}$ if $x = u + \frac{1}{u}$, $y = u - \frac{1}{u}$

Answers

Q2. $-\frac{am}{(ax+b)^{m+1}}$ Q5. $-\sqrt{\frac{y}{x}}$ Q6. $3x^2 + 2x + 2$

Q7. $7(6x+2)(3x^2+2x+9)^6$ Q8. $15x^2 - 14x + \frac{8}{x^2} - \frac{28}{x^5}$

Q9. $2b^3x + 9a^2x^2 + 15x^4$ Q10. $\frac{1-2x}{[(x-3)(x+2)]^2}$ Q11. $\frac{1-x^2}{(1+x^2)^2}$

Q12. $\frac{2}{(1-x)^2}$ Q13. -1 **Q14**. $n(3x^3 - 7x^2 + x + 1)^{n-1}(9x^2 - 14x + 1)$

Q16. $\frac{(y+3)^2}{7}$ Q17. $-\left(\frac{y}{x}\right)^{1/3}$ Q18. $-\frac{(ax+hy)}{(by+hx)}$

Q19. $\frac{1}{(1+x^3)^2}$ Q.20: $\frac{1}{(1+x^2)^2}$ Q21. $\frac{u^2+1}{u^2-1}$

Objective Type Questions

Q.1: Encircle the correct one, of the given answers in each item.

1. $\frac{d}{dx}(2x+3)^4 =$

[✓ (a)] $8(2x+3)^3$ (b) $4(2x+3)^3$

(c) $(2x+3)^3$ (d) $4(2x+3)^2$

S.P 2. $\frac{d}{dx}\left(\frac{1}{x}\right) =$

(a) $\frac{1}{x^2}$ [✓ (b)] $-\frac{1}{x^2}$

(c) $-\frac{1}{x^3}$ (d) $\frac{2}{x}$

3. $m\,x^{m-1}$ is the differential w.r.t. x of

[✓ (a)] $m(m-1)x^{m-2}$ (b) $(m-1)\,x^{m-2}$

(c) x^m (d) $m\,x^m$

4. $\frac{d}{dx}(ax+b)^2$

(a) $2(ax+b)$ [✓ (b)] $2a(ax+b)$

(c) $\frac{(ax+b)^3}{3}$ (d) $2\,(ax+b)\,b$

S.P 5. Second derivative of x^2 is

[✓ (a)] 2 (b) 2x

(c) Zero (d) $2\,x^2$

6. $\lim_{x\to 0}\frac{\delta y}{dx}$ is also denoted by

[✓ (a)] $\frac{dy}{dx}$ (b) $\int dx$

(c) y (d) $D^2\,y$

7. $\frac{dy}{dx}\sqrt{1+x})$

(a) $\frac{1}{\sqrt{1+x}}$ [✓ (b)] $\frac{1}{2\sqrt{1+x}}$

(c) $(1+x)^{1/3}$ (d) $\frac{-1}{2\sqrt{1+x}}$

8. If $u = t^2 - 3$ then $\frac{du}{dt} =$

[✓ (a)] $2t$ (b) $2t - 3$

(c) t^{-2} (d) $2t^{-2}$

9. If $y = u^2$ and $u = x$ then $\frac{dy}{dx}$

[✓ (a)] $2x$ (b) u^2

(c) x (d) $2x^2$

10. If $y = \frac{x+1}{x}$ then $\frac{dy}{dx} =$

[✓ (a)] $-\frac{1}{x^2}$ (b) $\frac{x+1}{x^2}$

(c) $\frac{2}{x^2}$ (d) $\frac{x^2 - 1}{x^2}$

Answers

Q.1:	i.	a	ii.	b	iii.	a	iv.	b	v.	a
	vi.	a	vii.	b	viii.	a	ix.	a	x.	a

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Chapter 3

Differentiation of Trigonometric and Inverse Trigonometric Functions

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3.1 Differentiation of Trigonometric Functions.

3.1.1 Derivative of sin x:

$$\frac{d}{dx}(\sin x) = \cos x$$

Proof : Let $y = \sin x$

$$y + \delta y = \sin(x + \delta x)$$

$$\delta y = \sin(x + \delta x) - \sin x$$

Using the Formula $\mathit{Sin}\, a - \mathit{Sin}\, b = 2\, \mathit{Cos}\left(\frac{a+b}{2}\right)\ \mathit{Sin}\left(\frac{a-b}{2}\right)$ *we have*

$$\delta y = 2\cos\frac{1}{2}(2x + \delta x)\ \sin\frac{\delta x}{2}$$

$$\delta y = 2\cos\left(x + \frac{\delta x}{2}\right)\ \sin\frac{\delta x}{2}$$

$$\frac{\delta y}{\delta x} = \frac{2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$\lim_{x\to 0}\ \frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\ \frac{2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x\to 0}\ \cos\left(x + \frac{\delta x}{2}\right)\ \lim_{\delta x\to 0}\ \frac{\sin\frac{\delta x}{2}}{\left(\frac{\delta x}{2}\right)}$$

So, $\frac{dy}{dx} = \cos x . 1 = \cos x.$ $\therefore \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$, where θ is in radian.

$$\therefore \boxed{\frac{d}{dx}(\sin x) = \cos x.}$$

3.1.2 Derivative of cos x:

$$\frac{d}{dx}(\cos x) = -\sin x$$

Proof : Let $y = \cos x$

$$y + \delta y = \cos(x + \delta x)$$

$$\delta y = \cos(x + \delta x) - \cos x$$

Using the Formula $Cosa - Cosb = -2\, Sin\, \frac{a+b}{2}\;\; Sin\, \frac{a-b}{2}$, *we have*

$$\delta y = -2 \text{ sins } \frac{1}{2}(2x + \delta x) \;\; \sin\frac{\delta x}{2}$$

$$\delta y = -2 \sin\left(x + \frac{\delta x}{2}\right) \sin\frac{\delta x}{2}$$

$$\frac{\delta y}{\delta x} = \frac{-2\sin\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$\lim_{x\to 0}\frac{\delta y}{\delta x} = \lim_{x\to 0} -2\,\frac{\sin\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$\frac{dy}{dx} = -\lim_{\delta x\to 0}\sin\left(x + \frac{\delta x}{2}\right)\lim_{\delta x\to 0}\frac{\sin\frac{\delta x}{2}}{\left(\frac{\delta x}{2}\right)}$$

$$\frac{dy}{dx} = -\sin x.1 = -\sin x$$

$$\therefore \boxed{\frac{d}{dx}(\cos x) = -\text{Sin } x}$$

3.1.3 Derivative of tan x:

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Proof : Let $y = \tan x$

$$y + \delta y = \tan(x + \delta x)$$

$$y + \delta y - y = \tan(x + \delta x) - \tan x$$

$$\delta y = \tan(x + \delta x) - \tan x$$

$$\delta y = \frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x}$$

$$\delta y = \frac{\sin(x+\delta x)\cos x - \cos(x+\delta x)\sin x}{\cos(x+\delta x)\cos x}$$

[As, $\sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha - \beta)$]

$$\delta y = \frac{\sin(x + \delta x - x)}{\cos(x + \delta x)\cos x}$$

$$\frac{\delta y}{\delta x} = \frac{\sin \delta x}{\delta x \cos(x + \delta x)\cos x}$$

$$\lim_{x \to 0} \frac{\delta y}{\delta x} = \lim_{x \to 0} \frac{\sin \delta x}{\delta x \cos(x + \delta x)\cos x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} \lim_{\delta x \to 0} \frac{1}{\cos(x + \delta x)\cos x}$$

$$\frac{dy}{dx} = 1 . \frac{1}{\cos x \cos x} = \frac{1}{\cos^2 x}$$

$$\therefore \quad \frac{dy}{dx} = \sec^2 x$$

$$\therefore \quad \boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

3.1.4 Derivative of cot x:

$$\boxed{\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x}$$

3.1.5 Derivative of Sec x:

$$\boxed{\frac{d}{dx}(\sec x) = \sec x . \tan x}$$

3.1.6 Derivative of cosec x:

$$\boxed{\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x . \cot x}$$

Example 1:

Differentiate the following by first principle : $\sin 2x$

Solution :

$$y = \sin 2x$$

$$y + \delta y = \sin 2(x + \delta x)$$

$$y + \delta y - y = \sin 2(x + \delta x) - \sin 2x$$

$$\delta y = \sin 2(x + \delta x) - \sin 2x$$

$$\delta y = 2\cos\left(\frac{2x + 2\delta x + 2x}{2}\right) . \sin\left(\frac{2x + 2\delta x - 2x}{2}\right)$$

$$\delta y = 2\cos(2x + \delta x)\sin \delta x$$

$$\frac{\delta y}{\delta x} = \frac{2\cos(2x + \delta x)\sin \delta x}{\delta x}$$

$$\frac{\delta y}{\delta x} = 2\cos(2x + \delta x)\frac{\sin \delta x}{\delta x}$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 2 \lim_{\delta x \to 0} \cos(2x + \delta x) \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}$$

$$\frac{dy}{dx} = 2\cos 2x . 1$$

$$\frac{dy}{dx} = 2\cos 2x$$

Note: General Rule

Let $y = (\text{Trigonometric Function of angle})^n$

Then , $\frac{dy}{dx} = \frac{d}{dx}(\text{Trigonometric Function of angle})^n$

$= n\,(\text{Trig.fun. of angle})^{n-1}$ (derivative of Trig. Fun.).(derivative of angle)

Example 2:

i. $\cos^3 x$ ii. $\sqrt{\sin\sqrt{x}}$

Solution:

i. $y = \cos^3 x.$

$$\frac{dy}{dx} = 3\cos^2 x \frac{d}{dx}(\cos x)$$

$$= 3\cos^2 x(-\sin x)$$

$$= -3\cos^2 x \sin x$$

(ii) $y = (\sin\sqrt{x})^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(\sin\sqrt{x})^{-\frac{1}{2}} \frac{d}{dx}(\sin\sqrt{x})$$

$$= \frac{1}{2}(\sin\sqrt{x})^{-\frac{1}{2}} \cos\sqrt{x} \frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{2\sqrt{\sin\sqrt{x}}} \cos\sqrt{x} \, \frac{1}{2}(x)^{-1/2}$$

$$= \frac{\cos\sqrt{x}}{4\sqrt{x}\sqrt{\sin\sqrt{x}}}$$

Example 3:

Find the derivative of

$\sqrt{x+1}\tan^3 2x$

Solution:

i. Let $y = \sqrt{x+1}.\tan^3 2x$

Then $\frac{dy}{dx} = \sqrt{x+1}\frac{d}{dx}(\tan^3 2x) + \tan^3 2x \frac{dy}{dx}(\sqrt{x+1})$

$= \sqrt{x+1}\,.\,3\tan^2 2x \frac{d}{dx}(\tan 2x) + \tan^3 2x \frac{1}{2}(x+1)^{-½} \frac{d}{dx}(x + 1)$

$$= \sqrt{x+1}\,.\,3\tan^2 2x \sec^2 2x \frac{d}{dx}(2x) + \frac{1}{2}\tan^3 2x \frac{1}{\sqrt{x+1}} \quad (1)$$

$$= \sqrt{x+1}\,.\,3\tan^2 2x \sec^2 2x\, 2 + \frac{\tan^3 2x}{2\sqrt{x+1}}$$

$$= 6\sqrt{x+1}\ \tan^2 2x \sec^2 2x + \frac{\tan^3 2x}{2\sqrt{x+1}}$$

Example 4:

i. Find $\frac{dy}{dx}$ if $y - \cos(x+y) = 0$

ii. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots\infty}}}$

prove that $(2y-1)\frac{dy}{dx} = \cos x$

Solution: Let $y - \cos(x+y) = 0$

$y = \cos(x+y)$

i. Differentiate w.r.t. x.

$$\frac{dy}{dx} = -\sin(x+y)\frac{d}{dx}(x+y)$$

$$= -\sin(x+y)\left(1+\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = -\sin(x+y) - \sin(x+y)\frac{dy}{dx}$$

$$\frac{dy}{dx} + \sin(x+y)\cdot\frac{dy}{dx} = -\sin(x+y)$$

$$\frac{dy}{dx}[1+\sin(x+y)] = -\sin(x+y)$$

$$\frac{dy}{dx} = \left[\frac{-\sin(x+y)}{1+\sin(x+y)}\right]$$

ii. Let $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots\infty}}}$

Squaring both sides

$$y^2 = \sin x + \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots\infty}}}$$

$$y^2 = \sin x + y$$

Differentiate w.r.t.x.

$$2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$2y\frac{dy}{dx} - \frac{dy}{dx} = \cos x$$

$$(2y-1)\frac{dy}{dx} = \cos x$$

3.2 Differentiation of Parametric Equations:

i. Find $\frac{dy}{dx}$ for $\theta = \frac{\pi}{2}$ when $x = 2\cos\theta - \cos 2\theta$, $y = 2\sin\theta - \sin 2\theta$

ii. Find $\frac{dy}{dx}$ when $x = a\cos^3 t$, $y = b\sin^3 t$.

Solution:

i. We have

$$x = 2\cos\theta - \cos 2\theta$$

$$\frac{dx}{d\theta} = -2\sin\theta + 2\sin 2\theta$$

$$= -2(\sin\theta - 2\sin 2\theta)$$

$$y = 2\sin\theta - \sin 2\theta$$

$$\frac{dy}{d\theta} = 2\cos\theta - 2\cos\theta\,2 = 2(\cos\theta - \cos 2\theta)$$

By chain rule, $\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$

$$\frac{dy}{dx} = \frac{2(\cos\theta - \cos 2\theta)}{-2(\sin\theta - 2\sin 2\theta)}$$

At $\theta = \frac{\pi}{2}$, we get

$\frac{\pi}{2} = \frac{180}{2} = 90$

$$\left(\frac{dy}{dx}\right)_{\frac{\pi}{2}} = \frac{\cos\frac{\pi}{2} - \cos 2\frac{\pi}{2}}{-\left(\sin\frac{\pi}{2} - \sin 2\frac{\pi}{2}\right)} = \frac{0-(-1)}{-(1-0)}$$

$$\left(\frac{dy}{dx}\right)_{\frac{\pi}{2}} = -1$$

ii. $x = a\cos^3 t$

$$\frac{dy}{dt} = 3a\cos^2 t\,\frac{d}{dt}(\cos t)$$

$$= -3a\cos^2 t.\sin t$$

$$y = b\sin^3 t$$

$$\frac{dy}{dt} = 3b\sin^2 t.\,\frac{d}{dt}(\sin t)$$

$$= 3b\sin^2 t.\cos t$$

By chain rule: $\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx}$

$$\frac{dy}{dx} = 3b\sin^2 t.\,\text{cost}.\frac{1}{-3a\cos^2 t.\sin t}$$

$$\frac{dy}{dx} = \frac{3b\sin^2 t\cos t}{-3a\cos^2 t.\sin t}$$

$$\frac{dy}{dx} = \frac{b\sin t}{-a\cos t}$$

$$\frac{dy}{dx} = -\frac{b}{a}\tan t$$

3.3 Differentiation of a Function w.r.t. another Function:

Example 6:

i. Differentiate $\tan^2 x$ w.r.t. $\cot^3 x$.

ii. Differentiate $\cos x$ w.r.t. $\tan x$.

Solution:

i. Let $y = \tan^2 x$ and $t = \cot^3 x$.

$\frac{dy}{dx} = 2\tan x\sec^2 x$ | $\frac{dt}{dx} = 3\cot^2 x(-\text{cosec}^2 x)$

$$\frac{dt}{dx} = -3\cot^2 x \operatorname{cosec}^2 x$$

By chain Rule , $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.

$$\frac{dy}{dt} = 2\tan x \sec^2 x \cdot \frac{1}{-3\cot^2 x \operatorname{cosec}^2 x}$$

$\frac{2}{3}\frac{\sin x}{\cos x}\cdot\frac{\sin^2 x}{\cos^2 x}$

$$= \frac{2\tan x \sec^2 x}{-3\cot^2 x \operatorname{cosec}^2 x} = -\frac{2}{3}\frac{\sin x}{\cos x}\cdot\frac{\sin^2 x}{\cos^2 x}\cdot\frac{\sin^2 x}{\cos^2 x}$$

$$= -\frac{2}{3}\frac{\sin^5 x}{\cos^5 x}$$

$$\frac{dy}{dt} = \frac{2}{3}\tan^5 x$$

ii. Let $y = \cos x$ and $t = \tan x$

$\frac{dy}{dx} = -\sin x$ and $\frac{dt}{dx} = \sec^2 x$

By chain Rule , $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$$\frac{dy}{dt} = -\sin x \cdot \frac{1}{\sec^2 x}$$

$$\frac{dy}{dt} = -\sin x \cos^2 x.$$

Exercise 3.1

Q.1: Differentiate w.r.t x from first principle method.

(i) $\cot x$ (ii) $\operatorname{cosec} x$ (iii) $\cos 2x$

Q.2: Differentiate the following functions w.r.t.x.

(i) $\sin nx$ (ii) $\sin^n x$ (iii) $\sin x^n$

(iv) $\cos^2(ax + b)$ (v) $\sin\sqrt{x} + \sqrt{\sin x}$ (vi) $\operatorname{cosec}^2 3x$

(vii) $\sec\sqrt{a + bx}$ (viii) $\sin(\tan x)$ (ix) $\cos^3(3x + 1)$

(x) $\sin[\sin(\cos x)]$

Q.3: Find the derivative of

(i) $x^2 \tan x$ (ii) $x^2 \sec 4x$ (iii) $\sin^m x \sin mx$

(iv) $\sin^2 x \cos^3 x$ (v) $(ax+b)\sqrt{1+\sin 2x}$ (vi) $\dfrac{1+\tan x}{1-\tan x}$

(vii) $\dfrac{\sec 2x}{1+\tan 2x}$ (viii) $\sqrt{\dfrac{1-\cos x}{1+\cos x}}$ (ix) $\dfrac{\sin x}{1-\cos x}$

(x) $(x \sin x^2)^{4/3}$

Q.4: If $\sin y = x \sin(a+y)$, prove that $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$

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Q.5 (a) If $xy = \cos(x+y)$, Show that $\dfrac{dy}{dx} + \dfrac{y+\sin(x+y)}{x+\sin(x+y)} = 0$

(b) Find dy/dx if $y = x \cos y$

Q.6: If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots\dots\dots\infty}}}$

Prove that $(2y-1)\dfrac{dy}{dx} = \sec^2 x$

Q.7: (i) If $y = a\sin\theta + b\cos\theta$, Show that $y^2 + \left(\dfrac{dy}{d\theta}\right)^2 = a^2 + b^2$

(ii) If $x = a\cos^3\theta$, $y = b\sin^3\theta$, show that $a\dfrac{dy}{dx} + b\tan\theta = 0$

Q.8: Find $\dfrac{dy}{dx}$ When

(i) $x = a\sin t$, $y = \cos at$

(ii) $x = a(\cos t + \sin t)$, $y = a(\sin t - t\cos t)$

Q.9: Differentiate

(i) $\sin^3 x$ w.r.t. $\sin x^3$ (ii) $\cos x$ w.r.t. $\tan x$

(iii) $\sin^2 x$ w.r.t. $\cos^4 x$ (iv) $\sin x$ w.r.t. $\tan x$

Answers 3.1

Q.1: (i) $-\operatorname{cosec}^2 x$ (ii) $-\operatorname{cosec} x \cot x$ (iii) $-2\sin 2x$

Q.2: (i) $n\cos nx$ (ii) $n\sin^{n-1}x\cos x$. (iii) $n x^{n-1}\cos x^n$

(iv) $-a\sin 2(ax+b)$ (v) $\frac{\cos\sqrt{x}}{2\sqrt{x}}+\frac{\cos x}{2\sqrt{\sin x}}$

(vi) $-6\operatorname{cosec}^2 3x\cot 3x$ (vii) $\frac{b\sec\sqrt{a+bx}\tan\sqrt{a+bx}}{2\sqrt{a+bx}}$

(viii) $\cos(\tan x)\sec^2 x$ (ix) $-9\cot^2(3x+1)\operatorname{cosec}^2(3x+1)$

(x) $-\cos[\sin(\cos x)]\cos(\cos x)\sin x$

Q.3: (i) $x^2\sec^2 x+2x\tan x$ (ii) $2x\sec 4x\,(2x\tan 4x+1)$

(iii) $m\sin^{m-1}x\sin x\,(1+m)$

(iv) $\cos^2 x\sin x\,(2\cos^2 x-3\sin^2 x)$

(v) $\frac{(ax+b)(\cos 2x)+a(1+\sin 2x)}{\sqrt{1+\sin 2x}}$

(vi) $\frac{2}{1-\sin 2x}$

(vii) $2\sec 2x\,\frac{\tan 2x-1}{(\tan 2x+1)^2}$ (viii) $\frac{1}{2}\sec^2\frac{x}{2}$

(ix) $\frac{-1}{1-\cos x}$ (x) $\frac{4}{3}(x\sin x^2)^{1/3}(2x^2\cos x^2+\sin x^2)$

Q.5 (b) $\frac{\cos y}{1+x\sin y}$

Q.8: (i) $\frac{-\sin at}{\cos t}$ (ii) $\frac{t\sin t}{\cos t-\sin t}$

Q.9: (i) $\frac{\sin^2 x\cos x}{x^2\cos x^3}$ (ii) $-\sin x\cos^2 x$ (iii) $-\frac{1}{2}\sec^2 x$ (iv) $\cos^3 x$

3.4 Differentiation of Inverse Trigonometric Functions:

The functions of the form $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$ and $\text{cosec}^{-1}x$ are defined as the inverse trigonometric functions of the corresponding trigonometric functions sinx, cosx, tanx, cotx, secx and cosecx respectively.

$\text{Sin}^{-1}x$ is defined as the angle whose sine is equal to x. Thus $\text{Sin}^{-1}(1/2)$ is an angle whose sin is 1/2 and its least value is 30°, we know there are infinite values of x i.e. $(30^\circ + 2\pi n)$ for $y = 1/2$, but we only use the least value.

If $y = \sin^{-1}x$ then $x = \sin y$.

Note:

(i) $\tan^{-1}(\tan\theta) = \theta$

(ii) $\text{cosec}^{-1}\dfrac{1}{x} = \sin^{-1}x$

(iii) $\cot\left(\dfrac{4}{x}\right) = \tan^{-1}\dfrac{x}{4}$

(iv) $\sec^{-1}\dfrac{1}{2y} = \cos^{-1}2y$

(v) $\sin^{-1}x \neq (\sin x)^{-1}$, while $\dfrac{1}{\sin x} = (\sin x)^{-1}$

(vi) $\sin^{-1}x = \text{arc}\sin x$

3.4.1 Differentiation of $\sin^{-1}x$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

Proof: Let $y = \sin^{-1}x$

$$\text{Sin}\,y = x$$

$$x = \sin y$$

$$x + \delta x = \sin(y + \delta y)$$

$$\delta x = \sin(y + \delta y) - \sin y$$

$$\delta x = 2\cos\left(y + \frac{\delta y}{2}\right)\sin\frac{\delta y}{2}$$

$$\frac{\delta x}{\delta y} = \frac{2\cos\left(\frac{y+\delta y}{2}\right)\sin\frac{\delta y}{2}}{\delta y}$$

$$\lim_{\delta y\to 0}\frac{\delta x}{\delta y} = \lim_{\delta y\to 0}\cos\left(y+\frac{\delta y}{2}\right)\lim_{\delta y\to 0}\frac{\sin\frac{\delta y}{2}}{\frac{\delta y}{2}}$$

$$\frac{dx}{dy} = \cos y\ 1 = \cos y$$

$$\frac{dx}{dy} = \cos y \quad\Rightarrow\quad \frac{dy}{dx} = \frac{1}{\cos y}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

Note: (i) $\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

(ii) $\frac{d}{dx}(\sin^{-1} u)^n = n(\sin^{-1} x)^{n-1}\frac{d}{dx}(\sin^{-1} u)$

Note: We can also prove the above result as

$$y = \sin^{-1} x$$

$$\sin y = x$$

$$\frac{dx}{dy} = \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$$

Note: $\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

3.4.2 Differentiation of $\cos^{-1} x$

$$\boxed{\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}}$$

Proof : Let $y = \cos^{-1} x$

$$x = \cos y$$

$$x + \delta x = \cos(y + \delta y)$$

$$\delta x = \cos(y + \delta y) - \cos y$$

$$\delta x = \cos(y + \delta y) - \cos y$$

$$\delta x = -2\sin\left(y + \frac{\delta y}{2}\right)\sin\frac{\delta y}{2}$$

$$\frac{\delta x}{\delta y} = -\sin\left(y + \frac{\delta y}{2}\right)\frac{\sin\frac{\delta y}{2}}{\frac{\delta y}{2}}$$

$$\lim_{\delta y \to 0}\frac{\delta x}{\delta y} = -\lim_{\delta y \to 0}\sin\left(y + \frac{\delta y}{2}\right)\lim_{\delta y \to 0}\frac{\sin\frac{\delta y}{2}}{\frac{\delta y}{2}}$$

$$\frac{dx}{dy} = -\sin y \,.\, 1$$

$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1 - \cos^2 y}} = -\frac{1}{\sqrt{1 - x^2}}$$

$$\boxed{\frac{d}{dx}(\text{Cos}^{-1} x) = \frac{-1}{\sqrt{1 - x^2}}}$$

Note: $$\frac{d}{dx}(\text{Cos}^{-1} u) = \frac{-1}{\sqrt{1 - u^2}}\frac{du}{dx}$$

3.4.3 Differentiation of $\tan^{-1} x$

$$\boxed{\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}}$$

Proof : Let $y = \tan^{-1} x$

$$x = \tan y$$

$$x + \delta x = \tan(y + \delta y)$$

$$\delta x = \tan(y + \delta y) - \sin y$$

$$\delta x = \frac{\sin(y + \delta y)}{\cos(y + \delta y)} - \frac{\sin y}{\cos y}$$

$$\delta x = \frac{\sin(y + \delta y)\cos y - \sin y\cos(y + \delta y)}{\cos(y + \delta y)\cos y}$$

$$\delta x = \frac{\sin[(y + \delta y) - y]}{\cos(y + \delta y)\cos y}$$

$$\frac{\delta x}{\delta y} = \frac{\sin \delta y}{\delta y \cos(y + \delta y)\cos y}$$

$$\lim_{\delta y \to 0}\frac{\delta x}{\delta y} = \lim_{\delta y \to 0}\frac{\sin \delta y}{\delta y} \cdot \lim_{\delta y \to 0}\frac{1}{\cos(y + \delta y)\cos y}$$

$$= 1 \cdot \frac{1}{\cos y \cdot \cos y}$$

$$\frac{dx}{dy} = \frac{1}{(\cos y)^2} = \sec^2 y$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$= \frac{1}{1 + \tan^2 y}$$

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$$

Note: $$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1 + u^2}\,\frac{du}{dx}$$

3.4.4 Differentiation of $\cot^{-1} x$

$$\boxed{\frac{d}{dy}(\cot^{-1} x) = \frac{-1}{1 + x^2}}$$

Proof : Let $y = \cot^{-1} x$

$$x = \cot y$$

$$\frac{dx}{dy} = -\operatorname{cosec}^2 y = -(1 + \cot^2 y) = -(1 + x^2)$$

$$\frac{dx}{dy} = -(1 + x^2)$$

$$\frac{dy}{dx} = -\frac{1}{1 + x^2}$$

$$\frac{d}{dy}(\cot^{-1} x) = \frac{-1}{1 + x^2}$$

Note: $$\frac{d}{dy}(\cot^{-1} u) = \frac{-1}{1 + u^2}\frac{du}{dx}$$

You can also prove the above result by first principle.

3.4.5 Differentiation of $\sec^{-1}$ x.

$$\boxed{\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2 - 1}}}$$

Proof : Let $y = \sec^{-1} x$

$$x = \sec y$$

$$\frac{dx}{dy} = \sec y \tan y$$

$$\frac{dx}{dy} = \sec y\sqrt{\sec^2 y - 1}$$

$$\frac{dx}{dy} = x\sqrt{x^2 - 1}$$

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2 - 1}}$$

Note: $$\frac{d}{dx}(\sec^{-1}u) = \frac{-1}{u\sqrt{u^2-1}}\frac{du}{dx}$$

3.4.6 Differentiation of cosec⁻¹ x

$$\frac{d}{dx}(\operatorname{cosec}^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

Proof : Let $y = \operatorname{cosec}^{-1}x$

$$x = \operatorname{cosec} y$$

$$\frac{dx}{dy} = -\operatorname{cosec} y \cot y$$

$$\frac{dx}{dy} = -\operatorname{cosec} y\sqrt{\operatorname{cosec}^2 y - 1}$$

$$\frac{dx}{dy} = -x\sqrt{x^2-1}$$

$$\frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{cosec}^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

Note: $$\frac{d}{dx}(\operatorname{cosec}^{-1}u) = \frac{-1}{u\sqrt{u^2-1}}\frac{du}{dx}$$

Example 1:

Differentiation w.r.t. x

(i) $y = \sin^{-1}\sqrt{x}$ (ii) $y = \sin^{-1}\frac{x}{3}$

Solution:

(i) $y = \sin^{-1}\sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{\sqrt{1-x}} \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

(ii) $y = \sin^{-1}\frac{x}{3}$

Solution:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}} \frac{d}{dx}\left(\frac{x}{3}\right)$$

$$= \frac{1}{\sqrt{1-\frac{x^2}{9}}} \cdot \frac{1}{3}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{9-x^2}{9}}} \frac{1}{3} = \frac{3}{\sqrt{9-x^2}} \cdot \frac{1}{3}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{9-x^2}}$$

Example 2:

Differentiate w.r.t. x If $y = x^2 \cot^{-1} x$

Solution:

$$y = x^2 \cot^{-1} x$$

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\cot^{-1} x) + \cot^{-1} x \frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = x^2 \frac{(-1)}{1+x^2} + (\cot^{-1} x)\ (2x)$$

$$\frac{dy}{dx} = \frac{-x^2}{1+x^2} + 2x \cot^{-1} x$$

Example 3:

Find $\frac{dy}{dx}$ if $y = \cot^{-1}\frac{2x}{1-x^2}$

Solution:

$$y = \cot^{-1}\frac{2x}{1-x^2}$$

$$\frac{dy}{dx} = \frac{-1}{1+\left(\frac{2x}{1-x^2}\right)^2}\,\frac{d}{dx}\left(\frac{2x}{1-x^2}\right)$$

$$= \frac{-(1-x^2)^2}{(1-x^2)^2+(2x)^2}\left[\frac{(1-x^2)(2)-(2x)(-2x)}{(1-x^2)^2}\right]$$

$$= \frac{-(1-x^2)^2}{1-2x^2+x^4+4x^2}\left[\frac{2-2x^2+4x^2}{(1-x^2)^2}\right]$$

$$= \frac{-(2+2x^2)}{(1+2x^2+x^4)} = \frac{-2(1+x^2)}{(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{-2}{(1+x^2)}$$

Example 4:

If $y = \tan\left(2\tan^{-1}\frac{x}{2}\right)$ prove that

$$\frac{dy}{dx} = 4\,\frac{1+y^2}{4+x^2}$$

Solution:

$$\tan^{-1}y = 2\tan^{-1}\frac{x}{2}$$

$$\frac{1}{1+y^2}\,\frac{dy}{dx} = 2\,\frac{1}{1+\frac{x^2}{4}}\cdot\frac{1}{2}$$

$$\frac{1}{1+y^2}\,\frac{dy}{dx} = 2\,\frac{4}{4+x^2}\cdot\frac{1}{2}$$

$4\,\frac{1+y^2}{4+x^2}$

$$\frac{dy}{dx} = \frac{4(1+y^2)}{4+x^2}$$

Example 5:

Find $\frac{d}{dx}(\sin^{-1}\sqrt{1-x^2})$

Solution: $\frac{dy}{dx} = \frac{1}{\sqrt{1-(1-x^2)}} \cdot \frac{d}{dx}(\sqrt{1-x^2})$

$$= \frac{1}{\sqrt{x^2}} \; \frac{1}{2\sqrt{1-x^2}} \; \frac{d}{dx}(1-x^2)$$

$$= \frac{1}{x} \cdot \frac{-2x}{2\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

Alternative method:

Let $y = \sin^{-1}\sqrt{1-x^2}$, Put $x = \cos\theta$

Then, $y = \sin^{-1}\sqrt{1-\cos^2\theta}$

$y = \sin^{-1}\sqrt{\sin^2\theta}$

$y = \sin^{-1}(\sin\theta)$

$y = \theta$

$\frac{dy}{d\theta} = 1$

$x = \cos\theta$

$\frac{dx}{d\theta} = -\sin\theta$

By chain rule , $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$$\frac{dy}{dx} = 1\left(\frac{-1}{\sin\theta}\right) = \frac{-1}{\sin\theta}$$

$$= -\frac{1}{\sqrt{\sin^2\theta}} = -\frac{1}{\sqrt{1-\cos^2\theta}}$$

$$\therefore \quad \frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)=\frac{-1}{\sqrt{1-x^2}}$$

Example 6: Differentiate $\tan^{-1}\sqrt{1-x^2}$ w.r.t. $\cos^{-1}x$

Solution:

Let $y=\tan^{-1}\sqrt{1-x^2}$ and $u=\cos^{-1}x$

Differentiate both equations w.r.t x

$$\frac{dy}{dx}=\frac{1}{1+(\sqrt{1-x^2})^2}\,\frac{d}{dx}\sqrt{1-x^2}; \qquad \frac{du}{dx}=-\frac{1}{\sqrt{1-x^2}}$$

$$=\frac{1}{1+1-x^2}\cdot\frac{1}{2\sqrt{1-x^2}}(-2x)$$

$$=\frac{1}{2-x^2}\,\frac{-x}{\sqrt{1-x^2}}=\frac{-x}{(2-x^2)\sqrt{1-x^2}}$$

By chain rule, $\dfrac{dy}{du}=\dfrac{dy}{dx}\cdot\dfrac{dx}{du}$

$$\frac{dy}{dx}=\frac{-x}{(2-x^2)\sqrt{1-x^2}}\cdot-\frac{\sqrt{1-x^2}}{1}=\frac{x}{2-x^2}$$

Exercise 3.2

Q.1: Differentiate w.r.t. x.

(i) $\tan^{-1}\sqrt{x}$ (ii) $\tan^{-1}\dfrac{1}{x^2}$ (iii) $y=\sin^{-1}x^2$

(iv) $\sec^{-1}(\sin x)$ (v) $\tan^{-1}(1/x)$ (vi) $\cos^{-1}(2x)$

Q.2: Differentiate each of the following expression w.r.t. x.

(i) $\tan x.\sin^{-1}x$ (ii) $\tan^{-1}\dfrac{x-1}{x+1}$ (iii) $\tan^{-1}(\sec x+\tan x)$

(iv) $\sec^{-1}\left(\dfrac{x^2+1}{x^2-1}\right)$ (v) $x^2\sec^{-1}5x$

Q.3: If $y=\tan(p\tan^{-1}x)$, show that $(1+x^2)\dfrac{dy}{dx}=p(1+y^2)$

Q.4: Solve the following:

V. Imp

(i) If $y = x \tan^{-1}\frac{x}{y}$ then prove that $\frac{dy}{dx} = \frac{y}{x}$

(ii) If $y = \sin^{-1}\frac{2x}{1+x^2}$ then prove that $\frac{dy}{dx} = \frac{2}{1+x^2}$

Q.5: Differentiate

(i) $\sin^{-1}x$ w.r.t. $\cos^{-1}x$

(ii) $\cos^{-1}\sqrt{x}$ w.r.t. $\sqrt{1-x}$

Answers 3.2

Q.1: (i) $\frac{1}{2\sqrt{x}(1+x)}$ (ii) $\frac{-2x}{1+x^4}$ (iii) $\frac{2x}{\sqrt{1-x^4}}$

(iv) $\frac{\cot x}{\sqrt{\sin^2 x-1}}$ (v) $\frac{-1}{x^2+1}$ (vi) $\frac{-2}{\sqrt{1-4x^2}}$

Q.2:

(i) $\sec^2 x.\sin^{-1}x + \frac{\tan x}{\sqrt{1-x^2}}$ (ii) $\frac{1}{x^2+1}$ (iii) $\frac{1}{2}$

(iv) $\frac{-2}{1+x^2}$ (v) $\frac{x}{\sqrt{25x^2-1}} + 2x\sec^{-1}5x$

Q.5: (i) -1 (ii) $\frac{1}{\sqrt{x}}$

3.5 Differentiation of Logarithmic and Exponential Functions

Important Limits and Results:

A few limits and results used in finding the derivatives of logarithmic and exponential functions from first principles.

(i) $\underset{x\to 0}{\text{Lt}}(1+x)^{1/x} = e$ (ii) $\underset{x\to 0}{\text{Lt}}\frac{a^x-1}{x} = \log_e a$

(iii) $\underset{x\to 0}{\text{Lt}}\frac{e^x-1}{x} = 1$ (iv) $\log_e mn = \log_e m + \log_e n$

(v) $\log_e \frac{m}{n} = \log_e m - \log_e n$ (vi) $\log_e m^n = n\log_e m$

(vii) $\log_e e = 1$ (viii) $a^m . a^n = a^{m+n}$

(ix) $\frac{a^m}{a^n} = a^{m-n}$ (x) $(a^m)^n = a^{mn}$

(xi) Logarithm definition: If a, N and x be three numbers such that $a^x = N$, then x is said to be Logarithm of N to the base 'a' and is written as $\log_a N = x$ where a and N are +ve.

If base is 10 the logarithm are called common logarithms and when the base is e = 2.7182 the logarithms are called natural or napiercan logarithms.

Note: $\log_e x$ is generally written as $\ln x$. When no base is given, the base e is understood.

3.6 Differentiation of Logarithmic Functions

3.6.1 Differentiation of $\log_a x$:

Let $y = \log_a x$

Then $y + \delta y = \log_a (x+\delta x)$

$$\delta y = \log_a (x+\delta x) - \log_a x$$

$$\delta y = \log_a \frac{x+\delta x}{x}$$

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x} \log_a \left(1+\frac{\delta x}{x}\right)$$

$$\frac{\delta y}{\delta x} = \log_a \left(1+\frac{\delta x}{x}\right)^{\frac{1}{\delta x}}$$

$$\frac{\delta y}{\delta x} = \frac{1}{x} \log_a \left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{1}{x} \log_n \left(1+\frac{\delta x}{x}\right)^{x/\delta x}$$

$$\frac{dy}{dx} = \frac{1}{x} \log_a \lim_{\delta x \to 0} \left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

Since $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = \lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$

So, $\lim\limits_{\delta x\to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}} = e$

$$\frac{dy}{dx}=\frac{1}{x}\log_a e$$

$$\frac{d}{dx}(\log_a x)=\frac{1}{x\ \text{In}\ a}$$

$$\therefore \quad \frac{d}{dx}=(\log_a x)=\frac{1}{x\ \text{In}\ a}$$

Note: (i) $\frac{d}{dx}(\log_a u)=\frac{1}{u\ \ln a}\frac{du}{dx}$

(ii) $\frac{d}{dx}(\log_a u)^{n} = n(\log_a u)^{n-1}.\frac{d}{du}\log_a u$

3.6.2 Differentiation of In x:

Let $y = \text{In}\ x$

$$y+\delta y=\ln(x+\delta x)-\text{In}\ x$$

$$y+\delta y-y=\ln(x+\delta x)-\text{In}\ x$$

$$\delta y=\ln(x+\delta x)-\ln x$$

$$\delta y=\ln\left(\frac{x+\delta x}{x}\right)$$

$$\delta y=\ln\left(1+\frac{\delta x}{x}\right)$$

$$\frac{\delta y}{\delta x}=\frac{1}{\delta x}\ \ln\left(1+\frac{\delta x}{x}\right)$$

$$\frac{\delta y}{\delta x}=\ln\left(1+\frac{\delta x}{x}\right)^{\frac{1}{\delta x}}$$

$$\frac{\delta y}{\delta x}=\frac{1}{x}\ln\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\frac{1}{x}\ln\left[\lim_{\delta x\to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right]$$

Since $\lim_{\delta x\to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}=e$

$$\frac{dy}{dx}=\frac{1}{x}\ \text{In}\ [e]$$

$$\frac{dy}{dx}=\frac{1}{x}$$

$$\therefore\quad \frac{d}{dx}\ (\text{In}\ x)=\frac{1}{x}$$

3.6.3 Differentiation of a^x:

Let $y=a^x$

Taking In on both sides

In $\quad \text{In}\ y=\text{In}a^x$

$\text{In}\ y=x\ \text{Ina}$

Diff. Both sides w.r.t. x

$$\frac{1}{y}\ \frac{dy}{dx}=\text{In}\ a$$

$$\frac{dy}{dx}=y\ \text{In}\ a$$

$$\frac{dy}{dx}=a^x\ \text{In}\ a$$

Note: $\frac{d}{dx}\ (a^u)=a^u\ \text{lua}\ \frac{du}{dx}$

3.6.4 Differentiation of e^x

Let $y=e^x$

Taking ln on both sides

$\text{ln}y = \text{ln}e^x$

$= x\ \text{ln}e$

$\text{ln}y = x$ as $\text{ln}e = 1$

Diff. both sides w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = y = e^x$$

$$\frac{d}{dx}(e^x) = e^x$$

Note: $$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

Example 1:

Differentiate the following w.r.t. x.

(i) $\text{In}\sqrt{x}$ (ii) $\text{In} \sin x$ (iii) $\text{In}(\text{in } x)$ (iv) $1/\ln x$

Solution:

(i) Let $y = \text{In}\sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}\frac{d}{dx}\left(\sqrt{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}\ \frac{1}{2\sqrt{x}} = \frac{1}{2x}$$

(ii) Let $y = \text{In} \sin x$

$$\frac{dy}{dx} = \frac{1}{\sin x}\ \frac{d}{dx}(\sin x)$$

$$\frac{dy}{dx} = \frac{1}{\sin x}\cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

(iii)Let $y = \text{In}((\text{In } x)$

$$\frac{dy}{dx} = \frac{1}{\text{In}x}\ \frac{d}{dx}(\text{In } x)$$

$$\frac{dy}{dx} = \frac{1}{\text{In } x}\frac{1}{x}$$

$$\frac{dy}{dx}=\frac{1}{x\,\text{In}x}$$

(iv) let $y=\dfrac{1}{\ln x}$

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{\ln x}\right)=\frac{d}{dx}(\ln x)^{-1}$$

$$=-1\,(\ln x)^{-2}\,\frac{1}{x}=\frac{-1}{x\,(\ln x)^{2}}$$

Example 2:

(i) $\sin(\text{In}\tan x)$ (ii) $\text{In}\sqrt{\dfrac{x^2+1}{x^2-1}}$ (iii) $x\,\text{In}\,3x$

Solution:

(i) Let $y=\sin((\text{In}\tan x)$

$$\frac{dy}{dx}=\cos(\text{In}\tan x)\frac{d}{dx}(\text{In}\tan x)$$

$$=\cos((\text{In}\tan x)\,\frac{1}{\tan x}\,\frac{d}{dx}(\tan x)$$

$$=\cos((\text{In}\tan x)\,\frac{1}{\tan x}\,\sec^2 x$$

$$=\frac{\cos(\text{In}\tan x)\sec^2 x}{\tan x}$$

(ii) Let $y=\text{In}\sqrt{\dfrac{x^2+1}{x^2-1}}$

$$y=\frac{1}{2}\,\text{In}\left(\frac{x^2+1}{x^2-1}\right)=\frac{1}{2}[\text{In}(x^2+1)-\text{In}(x^2-1)]$$

$$\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{x^2+1}\,\frac{d}{dx}(x^2+1)-\frac{1}{x^2-1}\,\frac{d}{dx}(x^2-1)\right]$$

$$\frac{dy}{dx}=\frac{1}{2}\left[\frac{2x}{x^2+1}-\frac{2x}{x^2-1}\right]$$

$$\frac{dy}{dx}=\frac{2}{2}\left[\frac{x}{x^2+1}-\frac{x}{x^2-1}\right]$$

$$\frac{dy}{dx}=\left[\frac{x(x^2-1)-x(x^2+1)}{(x^2+1)(x^2-1)}\right]$$

$$\frac{dy}{dx}=\frac{\cancel{x^3}-x-\cancel{x^3}-x}{(x^2+1)(x^2-1)}=\frac{-2x}{(x^4-1)}$$

(iii) Let $y = x \text{ In } 3x$

$$\frac{dy}{dx}=x\frac{d}{dx}(\text{In } 3x)+\text{In } 3x\frac{dx}{dx}$$

$$\frac{dy}{dx}=x\frac{1}{3x}\;\frac{d}{dx}(3x)+\text{In } 3x$$

$$\frac{dy}{dx}=\frac{x}{3x}\;3\pm\text{In } 3x$$

$$\frac{dy}{dx}=1+\text{In } 3x$$

Example 3:

Find $\frac{dy}{dx}$ of

(i) 3^x (ii) $a^{\sin x}$ (iii) e^{x^2} (iv) $e^x . \text{In} x$

Let $y = 3^x$

$$\frac{dy}{dx}=3^x \text{ In} 3$$

Because $\frac{d}{dx}(a^x)=a^x \text{ In } a$

(ii) Let $y=a^{\sin x}$

$$\frac{dy}{dx}=a^{\sin x}\text{In } a.\frac{d}{dx}(\sin x)$$

$$=\text{In } a\; a^{\sin x}\;\cos x$$

(iii) Let $y=e^{x^2}$

$$\frac{dy}{dx}=e^{x^2}\frac{d}{dx}(x^2)$$

$$= e^{x^2}\,2x$$

$$= 2x\,e^{x^2}$$

(iv) Let $y = e^x \ln x$

$$\frac{dy}{dx} = e^x \frac{d}{dx}(\ln x) + (\ln x)\frac{d}{dx}(e^x)$$

$$= e^x \frac{1}{x} + (\ln x)\,e^x$$

$$= e^x\left(\frac{1}{x} + \ln x\right)$$

3.7 Logarithmic Differentiation

In differentiation problems when an expression involve so many terms or variable powers, the process of differentiation becomes very much complicated. In order to solve such difficult problems use of logarithm is very help full for us. Such problems are of three types.

(i) Constant power variable

(ii) Variable power variable

(iii) When there are so many multiple function.

Example 4:

Differentiate the following:

(i) 3^x (ii) $(\sin x)^x$ (iii) $\sqrt{\dfrac{1+x^2}{1-x^2}}$

Solution:

(i) Let $y = 3^x$

Taking In to both the sides.

$$\ln y = \ln 3^x$$

$$\ln y = x \ln 3$$

Differentiate w.r.t.x

$$\frac{1}{y}\frac{dy}{dx} = 1 \ln 3$$

$$\frac{dy}{dx} = y \ln 3$$

$$\frac{dy}{dx} = 3^x \text{ In } 3$$

(ii) Let $y = (\sin x)^x$

Taking In to both the sides.

$$\text{In } y = \text{In } (\sin x)^x$$

$$\text{In } y = x \text{ In } \sin x$$

Differentiate w.r.t. x.

$$\frac{1}{y} \frac{dy}{dx} = x \frac{d}{dx} \text{ In } (\sin x) + \text{In } \sin x \frac{d}{dx}(x)$$

$$\frac{1}{y} \frac{dy}{dx} = x \frac{1}{\sin x} \frac{d}{dx}(\sin x) + \text{In } \sin x$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{x}{\sin x} \text{ Cos } x + \text{In } \sin x$$

$$\frac{dy}{dx} = y\,[x \cot x + \text{In } \sin x] = (\sin x)^x\,[x \cot x + \text{In } \sin x]$$

(iii) Let $y = \dfrac{x\sqrt{1+x^2}}{\sqrt{1-x^2}}$

Taking In to both sides.

$$\text{In } y = \text{In } \frac{x\sqrt{1+x^2}}{\sqrt{1-x^2}}$$

$$\text{In } y = \text{In } x + \text{In } \sqrt{1+x^2} - \text{In } \sqrt{1-x^2}$$

$$\text{In } y = \text{in } x + \frac{1}{2} \text{ In } (1+x^2) - \frac{1}{2} \text{ In } (1-x^2)$$

Differentiate w.r.t. x.

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} + \frac{1}{2} \frac{1}{(1+x^2)}(2x) - \frac{1}{2} \frac{1}{1-x^2}(-2x)$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} + \frac{x}{1+x^2} + \frac{x}{1-x^2}$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} + x\left[\frac{1}{1+x^2} + \frac{1}{1-x^2}\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{x}+x\left[\frac{1-x^2+1+x^2}{(1+x^2)(1-x^2)}\right]=\frac{1}{x}+\underbrace{\frac{2x}{(1+x^2)(1-x^2)}}$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1-x^4+2x^2}{x(1+x^2)(1-x^2)}$$

$(1^2-x^2)(1^2+x^2)$

(1^4-x^4)

$$\frac{dy}{dx}=+y\left[\frac{1-x^4+2x^2}{x(1+x^2)(1-x^2)}\right]$$

$$\frac{dy}{dx}=\frac{x\sqrt{1+x^2}}{\sqrt{1-x^2}}\left[\frac{1-x^4+2x^2}{x(1+x^2)(1-x^2)}\right]$$

$$\frac{dy}{dx}=\frac{1+2x^2-x^4}{(1+x^2)^{1/2}(1-x^2)^{3/2}}$$

Exercise 3.3

Q.1: Differentiate the following w.r.t. x.

(i) $x\ \text{In}\,(x^2-4)$ (ii) $\text{In}(x+\sqrt{x^2-1})$ (iii) $\text{In}\frac{x}{\sqrt{1+x^2}}$

(iv) $\text{In}\frac{1+\sin x}{1-\sin x}$ (v) $\tan^{-1}(\text{In}\,x)$ (vi) $\text{In}\frac{x^2+x+1}{x^2-x+1}$

(vii) $\text{In}(\sec x+\tan x)$ (viii) $\frac{x}{\text{In}\,x}$ (ix) $\text{In}\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$

(x) $x\ell nx-x$ (xi) $\cos x+\ell n\tan\frac{x}{2}$ (xii) $\ln(x+\sqrt{x^2+a^2})$

Q.2: Find $\frac{dy}{dx}$ for the following.

(i) e^{x^2} (ii) $x\,e^x$ (iii) $e^{\sqrt{x+1}}$

(iv) $\text{In}\frac{e^x+1}{e^x-1}$ (v) $x.\sin^{-1}x$ (vi) $e^{ax}.\,\text{Sin}\,bx$

(vii) $e^{2\ln\tan 5x}$ (viii) $a^{\sin x}\tan^{-1}x$ (ix) $\dfrac{e^{x\tan x}}{\sqrt{x}}$ (x) $e^{\sin 2x}$

Q.3: Find the derivative of the following:

(i) x^x (ii) $(\tan x)^{\sin^{-1}x}$ (iii) $x^{\sin x}$

(iv) $(\sin^{-1}x)^x$ (v) $(\sin x)^{\sin x}$ (vi) $x^y = y^x$

(vii) $x^x + x^{\sin x}$ (viii) $\dfrac{x\sqrt{1+x^2}}{\sqrt{1-x^2}}$ (ix) $\dfrac{\sqrt{x^2-1}(x+2)^3}{(x^3+2)^{1/3}}$

Answers 3.3

Q.1: (i) $\dfrac{2x^2}{x^2-4}+\ell n(x^2-4)$ (ii) $\dfrac{1}{\sqrt{x^2-1}}$ (iii) $\dfrac{1}{x(1+x^2)}$

(iv) $2\sec x$ (v) $\dfrac{1}{x[1+(\ln x)^2]}$ (vi) $\dfrac{2(1-x^2)}{1+x^2+x^4}$

(vii) $\sec x$ (viii) $\dfrac{\ln x-1}{(\ln x)^2}$ (ix) $\sec x$

(x) ℓnx (xi) $-\sin x + \operatorname{cosec} x$ (xii) $\dfrac{1}{\sqrt{x^2+a^2}}$

Q.2: (i) $2xe^{x^2}$ (ii) $e^x(x+1)$ (iii) $\dfrac{e^{\sqrt{x+1}}}{2\sqrt{x+1}}$

(iv) $\dfrac{e^x}{e^x+1}-\dfrac{e^x}{e^x-1}$ or $\dfrac{-2e^x}{e^{2x}-1}$ (v) $x\left[\dfrac{1}{\sqrt{1-x^2}}+\dfrac{\sin^{-1}x}{x}\right]$

(vi) $e^{ax}(a\sin bx + b\cos bx)$ (vii) $10\tan 5x\sec^2 5x$

(viii) $a^{\sin x}\left[\dfrac{1}{1+x^2}+\ln a\cos x\tan^{-1}x\right]$

(ix) $\dfrac{e^{x\tan x}}{\sqrt{x}}\left(x\sec^2 x+\tan x-\dfrac{1}{2x}\right)$ (x) $2\cos 2x\ e^{\sin 2x}$

Q.3: (i) $x^x(1+\ln x)$ (ii) $(\tan x)^{\sin^{-1}x}\left[\dfrac{2\sin^{-1}x}{\sin 2x}+\dfrac{\ln\tan x}{\sqrt{1-x^2}}\right]$

(iii) $x^{\sin x}\left(\cos x \,\text{In}\, x + \dfrac{\sin x}{x}\right)$

(iv) $(\sin^{-1}x)^{x}\left[\text{In}\sin^{-1}x + \dfrac{x}{\sin^{-1}x\sqrt{1-x^{2}}}\right]$

(v) $\text{Cosx}\,(\sin x)^{\sin x}\,(1 + \ell n \sin x)$ (vi) $\dfrac{y}{x}\left[\dfrac{x\,\text{In}\, y - y}{y\,\text{In}\, x - x}\right]$

(vii) $x^{x}[1 + \text{In}\, x] + x^{\sin x}\left[\dfrac{\sin x}{x} + \cos x\,\text{In}\, x\right]$

(viii) $\dfrac{1+2x^{2}-x^{4}}{(1-x^{2})^{3/2}(1+x^{2})^{1/2}}$

(ix) $\dfrac{(x+2)^{2}(3x^{5}-2x^{3}+10x^{2}+4x-6)}{\sqrt{x^{2}-1}(x^{3}+2)^{4/3}}$

SHORT QUESTIONS

Write the short answers of the following questions:

Q.1: Find the value of $\frac{d}{dx}\left(\frac{1 - \cos x}{\sin x}\right)$.

Q.2: Differentiate $\cos^2 x$ w.r.t. $\sin^2 x$

Q.3: Find the derivative of $x \cot x$ w.r.t. x

Q.4: If $x = a \sec\theta$, $y = b \tan\theta$ then find $\frac{dy}{dx}$

Q.5: Find the derivative of $\sin^{-1}\left(\frac{x}{a}\right)$

Q.6: Find the value of $\frac{d}{dx}(\sin^{-1}x + \cos^{-1}x)$

Q.7: Find the value of $\frac{d}{dx}\sec^{-1}(\sqrt{x})$

Q.8: Find the value of $\frac{d}{dx}\cos^{-1}(1 - 2x^2)$

Q.9: Find the derivative of $(\sec^{-1}x)^3$

Q.10: Find the value of the derivative of $\tan^{-1}x + \tan^{-1}\frac{1}{x}$

Q.11: Find the derivative of $\cos(\cot x)$

Q.12: Find the derivative of $\frac{\tan x}{x^2}$.

Q.13: If $x = \sin 2t$, $y = 2\cos t$ then find $\frac{dy}{dx}$.

Q.14: If $y = \cos(\log x)$, Find $\frac{dy}{dx}$

Q.15: Find the derivative of $\log(\cos^2 x)$

Q.16: Find $\frac{d}{dx}(a^{x^2})$

Q.17: Find the differential co-efficient of $e^{\tan^{-1}x}$

Q.18: Find the derivative of $e^{-2\log x}$ w.r.t. x

Hint: If $y = e^{\log x}$, Then $y = x$

Because, since $\log y = \log(e^{\log x}) = \log x . \log e = \log x$

So, $y = x$

Q.19: Find the value of $\frac{d}{dx}[e^{3\log \cos x}]$

Q.20: Find $\frac{d}{dx}(e^{2x} \cos 2x)$

Q.21: Find the value of $\frac{d}{dx}(x^x)$

Answers

(1) $\frac{1}{2}\text{Sec}^2\frac{x}{2}$ (2) -1 (3) $-x\,\text{cosec}^2 x + \cot x$ (4) $\frac{b}{a}\ \text{cosec}\,\theta$

(5) $\frac{1}{\sqrt{a^2 - x^2}}$ (6) 0 (7) $\frac{1}{2x\sqrt{x-1}}$ (8) $\frac{2}{\sqrt{1-x^2}}$ (9) $\frac{3(\sec^{-1})^2}{x\sqrt{x^2-1}}$

(10) 0 (11) $\sin(\cot x)\,\text{cosec}^2$ (12) $\frac{x\sec^2 x - 2\tan x}{x^3}$

(13) $\frac{-\sin t}{\cos 2t}$ (14) $\frac{\sin(\log x)}{x}$ (15) $-2\tan x$ (16) $2x\,\ell na.\, a^{x^2}$

(17) $\frac{e^{\tan^{-1}x}}{1+x^2}$ (18) $-\frac{2}{x^3}$ (19) $-3\cos^2 x \sin x$

(20) $2e^{2x}(\cos 2x - \sin 2x)$ (21) $x^x(1 + \ell nx)$

Objective Type Questions

Q.1: Encircle the correct one, of the given answers in each item.

S.P 1. $\frac{d}{dx}\sin x^3 =$

(a) $\cos x^3$ (b) $-\cos x^3$ (c) $3x\cos x^3$ [✓ (d) $3x^2\cos x^3$]

2. $\frac{d}{dx}\cos\left(\frac{1}{x}\right)$

(a) $-\sin\left(\frac{1}{x}\right)$ (b) $\sin\left(\frac{1}{x}\right)$

[✓ (c) $\frac{1}{x^2}\sin\left(\frac{1}{x}\right)$] (d) $-\frac{1}{x^2}\sin\left(\frac{1}{x}\right)$

3. $\frac{d}{dx}(\sin ax + \cos ax) =$

(a) $a\cos x + a\sin x$ (b) $a\cos x - a\sin x$

(c) $\cos ax + \sin ax$ [✓ (d) $a\cos ax - a\sin ax$]

4. $\frac{d}{dx}\text{Cosec } 3x =$

(a) $-\text{Cosec } 3x \text{ Cot } 3x$ [✓ (b) $-3 \text{ Cosec } 3x \text{ Cot } 3x$]

(c) $\text{Cot } 3x$ (d) $\text{cosec } 3x$

S.P 5. $\frac{d}{dx}\tan x^2 =$

[✓ (a) $2x\,\text{Sec}^2 x^2$] (b) $\text{Sec}^2 x^2$ (c) $\text{Sec}x^2$ (d) $\tan x^2$

6. $\frac{d}{dx}\tan\left(\frac{1}{\sqrt{x}}\right)$

(a) $\text{Sec}^2\left(\frac{1}{\sqrt{x}}\right)$ [✓ (b) $\text{cosec}^2\left(\frac{1}{\sqrt{x}}\right)$]

(c) $\frac{3}{2\sqrt{x}}\text{Sec}^2\left(\frac{1}{\sqrt{x}}\right)$ (d) $-\frac{1}{2(x)^{3/2}}\text{Sec}^2\left(\frac{1}{\sqrt{x}}\right)$

S.P 7. $\frac{d}{dx}(2\cos 3x)$

(a) $6\sin 3x$ [✓ (b) $-6\sin 3x$]

(c) $-6\cos 3x$ (d) $6\sec 3x$

8. $\frac{d}{dx}\sin(\sqrt{x})$

(a) $\cos(\sqrt{x})$ (b) $-\cos(\sqrt{x})$

[✓ (c) $\frac{1}{2\sqrt{x}}\cos(\sqrt{x})$] (d) $-\frac{1}{2\sqrt{x}}\cos(\sqrt{x})$

9. $\frac{d}{dx}\sec 3x =$

[✓ (a) $3\sec 3x\tan 3x$] (b) $\sec 3x\tan 3x$

(c) $3\sec 3x \cot 3x$ (d) $-3\sec 3x \tan 3x$

10. $\frac{d}{dx}\sin^{-1}x =$

(a) $\frac{1}{\sqrt{x^2-1}}$ (b) $\frac{1}{\sin^{-1}x\sqrt{1-x^2}}$

(c) $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$ [✓ (d) $\frac{1}{\sqrt{1-x^2}}$]

11. $\frac{d}{dx}\sin^{-1}(\sqrt{x})$

(a) $\frac{1}{\sqrt{1-x^2}}$ (b) $\frac{1}{\sqrt{1+x^2}}$

[✓ (c) $\frac{1}{2\sqrt{x}}\,\frac{1}{(1-x)}$] (d) $\frac{1}{2\sqrt{x}}\,\frac{1}{(1+x)}$

12. $\frac{d}{dx}\sin^{-1}\left(\frac{1}{x}\right)$

(a) $\frac{1}{\sqrt{x^2-1}}$ (b) $\frac{1}{X\sqrt{x^2-1}}$

[✓ (c) $-\frac{1}{X\sqrt{x^2-1}}$] (d) $-\frac{1}{\sqrt{x^2-1}}$

13. $\frac{d}{dx}\sec^{-1}2x =$

[✓ (a) $\frac{1}{x\sqrt{4x^2-1}}$] (b) $\frac{1}{x\sqrt{4x^2-1}}$

(c) $\frac{1}{x\sqrt{x^2-1}}$ (d) $\frac{1}{2x\sqrt{x^2-1}}$

14. $\frac{d}{dx}\tan^{-1}x^2 =$

(a) $\frac{1}{1+x^2}$ (b) $\frac{1}{1-x^2}$

[✓ (c) $\frac{2x}{1+x^4}$] (d) $\frac{2x}{1-x^4}$

15. $\frac{d}{dx}\sin^{-1}\left(\frac{1}{x}\right)$

(a) $-\frac{1}{1+x^2}$ [✓ (b) $\frac{-1}{x\sqrt{x^2-1}}$]

(c) $\frac{x}{1+x^2}$ (d) $\frac{1}{1-x^2}$

16. $\frac{d}{dx}a^x =$

[✓ (a) $a^x \ln a$] (b) $x\,a^{x-1}$

(c) a^{x-1} (d) a^x

17. $\frac{d}{dx} \ell n \, \text{Sin}\, x =$

[✓ (a) $\text{Cot}\, x$] (b) $\frac{1}{\text{Sinx}} \ell n \, \text{Sinx}$

(c) $\ell n \, \text{Cosx}$ (d) $\tan x$

18. $\frac{d}{dx} e^{3x} =$

(a) e^{3x-1} (b) e^{n-1}

[✓ (c) $3e^{3x}$] (d) $3x\, e^{3x}$

19. $\frac{d}{dx} x^{a} =$

[✓ (a) ax^{a-1}] (b) x^{n-1}

(c) ax^{a} (d) x^{a}

20. $\frac{d}{dx} \ell n \, (x^2 + 1) =$

(a) $\frac{x}{x^2 + 1}$ [✓ (b) $\frac{2x}{x^2 + 1}$]

(c) $\ell n \, (2x + 1)$ (d) $2x \, \ell n \, (2x + 1)$

21. $\frac{d}{dx} e^{\sin x}$

(a) $e^{\cos x}$ [✓ (b) $\text{Cosx}\, e^{\text{Sinx}}$]

(c) $\text{Sinx}\, e^{\text{Sin-1}}$ (d) $\text{Sinx}\, e^{\text{Sinx}}$

22. $\frac{d}{dx} \ell n \, \sin^{-1} x =$

(a) $\frac{1}{\sqrt{x^2 - 1}}$ [✓ (b) $\frac{1}{\text{Sin}^{-1} x\sqrt{1 - x^2}}$]

(c) $\frac{\text{Sin}^{-1}x}{\sqrt{1 - x^2}}$ (d) $\frac{\cos^{-1} x}{\sqrt{1 - x^2}}$

Answers

Q.1:
(1) d (2) c (3) d (4) b (5) a
(6) b (7) b (8) c (9) a (10) d
(11) c (12) c (13) a (14) c (15) b
(16) a (17) a (18) c (19) a (20) b
(21) b (22) b

Chapter 4
Rate of change of variable

4.1 Expressions for velocity and acceleration

(a) Velocity:

A body moving in a straight line covers a distance x(=OA) in time t so that A is its position at the end t and after a little longer time $t + \delta t$ its position is B so that $OB = x + \delta x$.

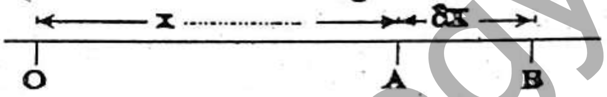


Fig.4.1

$\therefore AB = \delta x$, is the distance moved by the body in the small time δt.

Thus, average velocity of the body in this interval δt is given by $\frac{\delta x}{\delta t}$.

If $\delta t \to 0$, B will $\to$ A and the interval AB will become indefinitely small. In this case $\underset{\delta t \to 0}{Lt} \frac{\delta x}{\delta t}$ will give the velocity of the body at A.

Hence velocity at A $= \underset{\delta t \to 0}{Lt} \frac{\delta x}{\delta t} = \frac{dx}{dt}$

(b) Acceleration:

In fact the student has already learnt in the foregoing chapters that $\frac{dx}{dt}$ is the rate of change of x with regard to change in t and that exactly is the definition of velocity.

Thus if we have the relation between displacement x and time t given by the equation.

$X = f(t)$, then the velocity is given by $V = \frac{dx}{dt} = f'(t)$

In a similar manner if acceleration is denoted by 'a' we have

$a = \frac{dv}{dt}$ (acceleration is the rate of change of velocity with regard to change in time) Also $a = \frac{dv}{dx} = \frac{d}{dx}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = f''(t)$

Note: We can use X for the distance S i.e., $X = S = f(t)$
In case for free fall of body $X = 4.9t^2$

Example 1:

Motion of a particle along a straight line is given by $X = 8t^2$ where X is in meters and t in seconds. Find velocity and acceleration of the particle at t = 5sec and t = 10 second.

Solution:

$$X = 8t^2$$

Now velocity $V = \frac{dx}{dt} = 16t$

$\therefore \quad V = 16t$

(i) at $t = 5$ second
$V = 16 \times 5 = 80$ m/sec

(ii) at $t = 10$ second
$V = 16 \times 10 = 160$ m/sec

Acceleration $= a = \frac{dv}{dt}$

$$= a = \frac{d}{dt}(16t)$$

$$= 16 \text{ m/sec}^2$$

$\therefore$ Acceleration of the particle is constant at any instant of time

Example 2:

Figure shows the free fall of a heavy ball being released from rest at t = 0 sec.

(i) How many meters does the ball fall in the first two second?
(ii) What is its velocity and acceleration then?

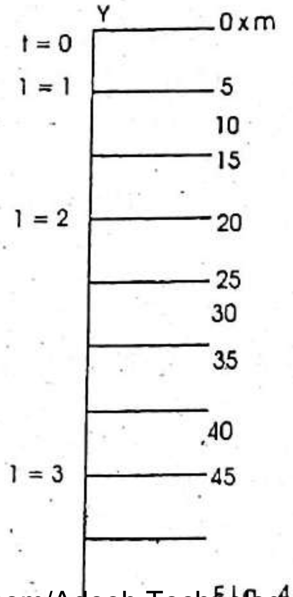


FIG. 4.2

Solution:

(i) The metric free fall equation is $X = 4.9t^2$ during the first two seconds, the ball falls $X(2) = 4.9(2)^2 = 19.6$m.

(ii) At any time t velocity is the derivative of displacement

$$V(t) = \frac{dx}{dt} = \frac{d}{dt}(4.9t^2)$$

$$V(t) = 9.8t$$

Velocity $= V(2) = 9.8 \times 2 = 19.6$ m/sec

The acceleration at any time t.

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(9.8t)$$

$$= 9.8 \text{ m/sec}^2$$

$\therefore$ At $t = 2$, the acceleration is 9.8 m/sec^2

4.2 Differentials / Slope of a line

In the preceding discussion of derivatives, $\frac{dy}{dx}$ is considered not as a fraction with numerator dy and denominator dx, but as a symbol denoting the limit of $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$. In some problems it is useful to interpret dx and dy separately. In this context dx referred to as the differential of x and dy as the differential of y. Differentials are useful, for example, in applications of integral calculus and in approximating changes in the dependent variable associated with small changes in the independent variable.

If $f'(x)$ is the derivative of $y = f(x)$ for a particular value of x and Δx is an increment of x, then the differential of $y = f(x)$, denoted by dy, is defined by the equation.

$$dy = f'(x)\,\Delta x = \frac{dy}{dx}\,\Delta x$$

If $f(x) = x$, then $f'(x) = 1$

and $dx = \Delta x$

Thus when x is the independent variable, the differential dx of x is equal to Δx.

So, $dy = f'(x)\,dx = \left(\frac{dy}{dx}\right) dx$

Thus the differential of a function or dependent variable equals its derivative multiplies by the differential of the independent variable.

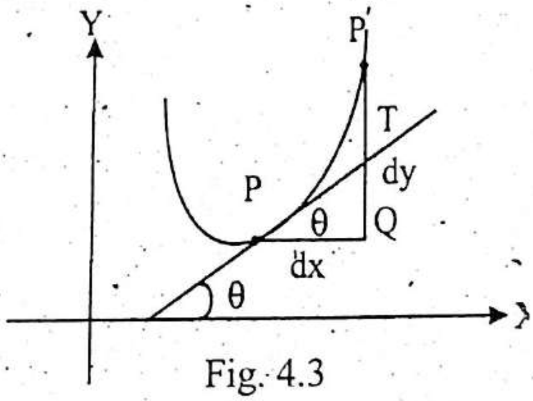


Fig. 4.3

Geometrically, consider the curve $y = f(x)$ (See Fig 4.6) and let $f(x)$ be the value of the derivative at P. Then

$dx = PQ$ and

$dy = f'(x)\,dx = (\tan\theta)(PQ)$

$$= \frac{QT}{PQ} \cdot PQ = QT.$$

dy denotes the increment of the ordinate of th tangent at P.

Note that the differential dy and the increment Δy of the function corresponding to the same value, $dx = \Delta x$ are not in general equal. In Fig 4.6, $dy = QT$ but $\Delta y = QP'$.

Another important application of differentials is made in calculating the approximate value that may arise in the value of say 'y' due to a small error made in the value of 'x' where $y = f(x)$.

Thus if δx is very small, δy will be small

$$\therefore \quad \delta y = \frac{dy}{dx}.\delta x \text{ approximately.}$$

This formula will enable us to find the error δy in y for an error δx in x. The approximate value is then given by $y + \delta y$ or $y + dy$.

4.3 Rate Measure

Whenever we are required to measure to rate of change of one quantity with regard to the change in the other, make use of derivative of one with respect to other.

e.g. The rate of change of y w.r.t. 'x' = $\frac{dy}{dx}$. Similarly rate of change of x w.r.t 't' is $\frac{dx}{dt}$. Hence derivative is used as a Rate Measure.

Example 3:

If $y = x^3 - 2x^2 + 9x + 7$, use differentials to find an approximate value of y when $x = 1.997$.

Solution:

Consider 1.997 to be the result of applying an increment $\Delta x = dx = -0.003$ to an original value of $x = 2$.
Then by differentiation.

$$\frac{dy}{dx} = 3x^2 - 4x + 9$$

$$dy = (3x^2 - 4x + 9)\,dx\,, \text{ when } x = 2 \text{ and } dx = -0.003$$

$$= (12 - 8 + 9)\,(-0.003)$$

$$dy = -0.039$$

Also, when $x = 2$, $y = 25$ (from given equation)

And -0.039 is the approximate change in y when x changes from 2 to 1.997.

Hence, $y + dy = 25 - 0.039 = 24.961$

And y is approximately 24.961 when x = 1.997

Example 4:

Using differentials, find an approximate value of $\sqrt{98}$

Solution:

The number 98 is close the number 100, which a perfect square. Thus the change in $y = \sqrt{x}$, corresponding to a change in x from 100 to 98, can be added to $y = \sqrt{100} = 10$ to obtain an approximate value for $\sqrt{98}$.

For $x = 100$ and $\Delta x = dx = -2$

But $y = \sqrt{x} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$

$$dy = \frac{dx}{2\sqrt{x}}$$

$$= \frac{-2}{2\sqrt{100}} = -0.1$$

and thus $\sqrt{98} = y + dy = \sqrt{100} - 0.1$

$= 9.9$ Approximately

Example 5:

Using differentials, find an approximate value for tan 46^0 given tan $45^0 = 1$, sec $45^0 = \sqrt{2}$, and $1^0 = 0.0175$.

Solution:

If $y = \tan x$,

$dy = \sec^2 x\, dx$

When x changes to dx, y changes to dy, approximately.

$x = 45^0 = \dfrac{\pi}{4}$ radians

$\Delta x = dx = 1^0 = 0.0175$

$dy = \sec^2 x\, dx = \left(\sqrt{2}\right)^2 (0.0175)$

$dy = 0.0350$

Since $y = \tan x = \tan 45^0 = 1$,

$$y + dy = 1 + 0.0350 = 1.0350$$

and tan $46^0 = 1.0350$ approximately.

Exercise 4.1

Q.1: The distance x meters moved by a point in t seconds is given by $x = t^3 + 3t^2 + 4$. Find the velocity and acceleration after 3 second.

Q.2: The velocity v m/s of a point moving in a straight line is given after t second by $v = 3t^2 + 4t$.

Find the acceleration after 2 second.

Q.3: The velocity v m/s of a point moving along a straight line, is given by $v = 4 - t^2$. Where t is the time in seconds.

Find the acceleration of the point when it is initially at rest.

Q.4: The distance moved by a point along a line in t second is given in meters by $x = t^2 + 4t$.

Find (i) The initial velocity

(ii) The velocity after 2 sec.

(iii) The average velocity for the first 2 sec.

Q.5: The distance x meter gone by a body in t second is given by $x = t^3 - 6t^2 + 4t$.

Find (i) Its initial velocity.

(ii) Its velocity after 2 seconds.

(iii) Its acceleration after 2 seconds.

Q.6: Use differentials to find the approximate value of each of the following:

(i) $\sqrt{65}$ (ii) $\sqrt[3]{124}$

Q.7: Find the rate of increase of the area of a square when the length of the side is 2cm and in increasing at 0.3 cm/s

Q.8: Find the rate of increase of the surface area of a sphere when the radius is 4cm and is increasing at 0.1 cm/s

Q.9: Find the rate of increase of the volume of a cube when the length of side is 2 cm and the area of a face is increasing at 1 cm^2/s.

Q.10: A container is made in the form of a 10cm cube to hold 1 litre (1000 cu.cm). How accurately must the inner edge be made so that the volume will be correct to within 3cubic centimeter(cu.cm)?

Answers 4.1

Q.1. 45 m/s , 24 m/s^2 Q.2. 16 m/s^2

Q.3. – 4m/s^2 (retardation) Q.4. (i) 4 m/s (ii) 8 m/s (iii) 6 m/s

Q.5. (i) 4 m/s (ii) –8 m/s (iii) 0 m/sec^2.

Q.6. (i) 8.0625 (ii) 4.9984 Q.7. 1.2 cm^2/s

Q.8. 3.2π cm^2/s or 10.023 cm^2/sec

Q.9. 3 cm^3/s Q.10. Error must not exceed 0.01 cm

4.5 Maxima and minima

4.5.1 Introduction

In this section we shall consider the application of calculus which maximize or minimize some aspect of something, like what is the size of the most profitable production run? What is the least expensive shape for an oil can? What is the stiffest beam we can cut from a 12 in long? In the mathematical models in which we use function to describe the things that benefit us, we usually answer such questions by findings maximum or minimum values of a differentiable function.

4.5.2 Increasing and decreasing function

Increasing function:

A function $f(x)$ is said to be an increasing function of x if as x increase $f(x)$ also increases or if as x decreases $f(x)$ also decreases.

Decreasing function:

A function $f(x)$ is said to be decreasing function of x. If as x increases $f(x)$ decreases or if as x decrease $f(x)$ increases.

Increasing and decreasing Functions by First Derivative:

Consider the function $y = f(x)$ at the point x = a. The first derivative of y with respect to x, $y' = f'(x)$ is the slope at the point x of the curve $y = f(x)$, in particular, $f'(a)$ is the slope of the curve $y = f(x)$ at the point x = a. Thus

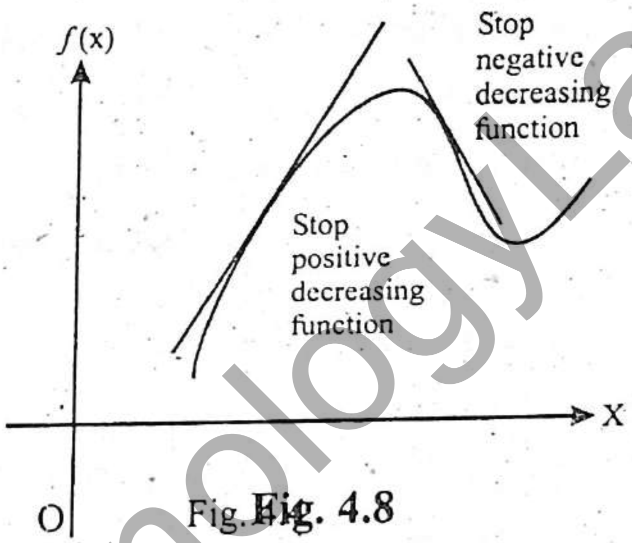


Fig. **Fig. 4.8**

1. If the first derivative $f'(a)$ is positive, $y = f(x)$ is an increasing function of x at x = a.
2. If the first derivative $f'(a)$ is negative, $y = f(x)$ is a decreasing function of x at x = a.
3. If the first derivative $f'(a)$ is neither positive nor negative at x = a, i.e. $f'(a) = 0$ then x = a is called stationary point or critical point. Obviously at that point tangent is parallel to x – axis. Also function is neither increasing nor decreasing at that point.

4.6 Relative (or Local) Maxima and Minima of a Function:

A function is said to have a relative (or local) Maximum at x = a if $f(a)$ is greater than any value of $f(x)$ for x in an interval around a, a function $y = f(x)$ is said to have a relative (or local) minimum at x = a if $f(x)$ is smaller than any value of $f(x)$ for x in an interval around a. Note that a relative maximum or minimum of a function is not its maximum or minimum for a given interval. Also note that it is possible for a relative maximum value of function to be less than a relative minimum value of the function **(see Fg.4.9).**

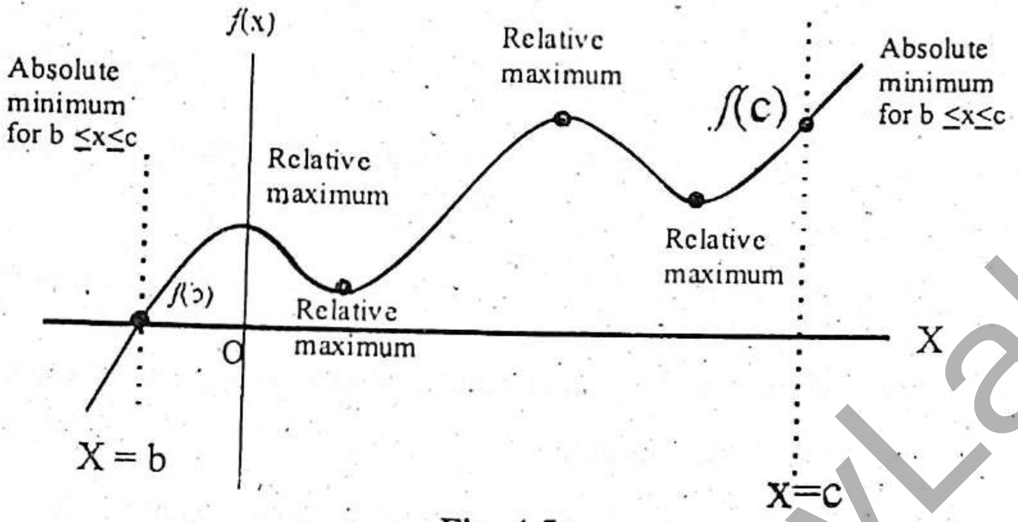


Fig. 4.5

If function f(x) has a relative maximum or minimum at a value x = *a* for which its first derivative is continuous, then $f'(a) = 0$.

4.6.1 Criteria for finding maxima and minima value by first derivative.

(It is also called near – point test).

Consider a function $f(x)$ at a value x = a for which $f(x)$ and $f'(x)$ are continuous. It is evident geometrically that if $f(a)$ is a relative maximum of $f(x)$, then the slope of $f(x)$ [that is, $f'(x)$] changes from positive to negative as x passes through the point x = a, similarly, if $f(a)$ is a relative minimum of $f(x)$, then slope of $f(x)$ [that is, $f'(x)$ changes from negative to positive as x passes through the point x = a **(see Fig 4.10).** The corresponding algebraic argument is based on the fact that an increasing function has a positive slope and a decreasing function has a negative slope.

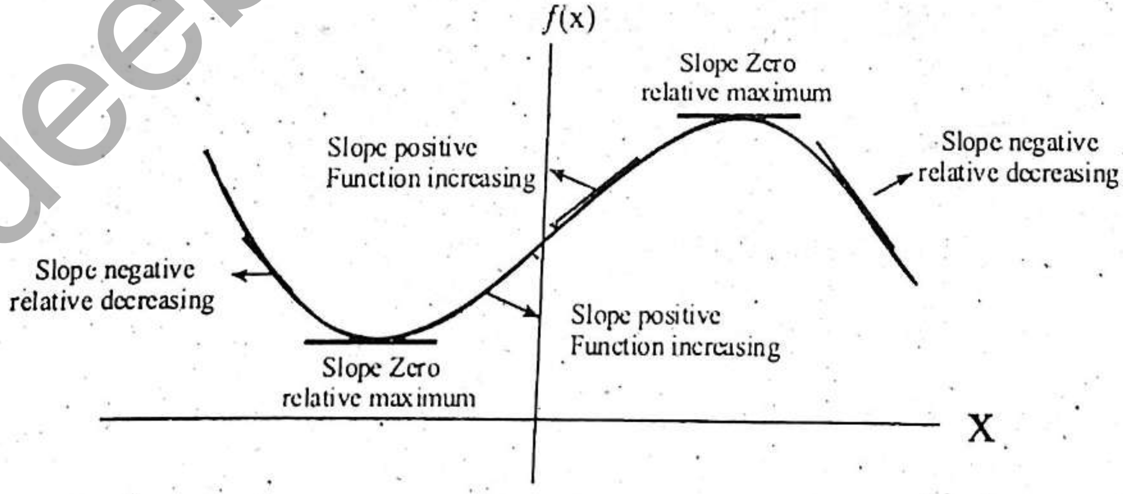

f(x)
Slope Zero relative maximum
Slope positive Function increasing
Slope negative relative decreasing
Slope negative relative decreasing
Slope positive Function increasing
Slope Zero relative maximum
X

Fig.4.10

If $f'(x)$ does not change sign as x passes through the point x = a, then x = a is called point of inflection. Tangent drawn at the point of inflection crosses the curve.

Thus, in order to determine the relative maximum and minimum values of a function $y = f(x)$.

1. Solve the equation $f'(x) = 0$ to obtain its roots or critical point (or turning points).
2. For each root a (or critical point), determine whether $f'(x)$ changes sign as x passes through a.

$f'(x)$ changes from + ve to –ve at x = a ⇒ relative maximum at x = a

$f'(x)$ changes from – ve to +ve at x = a ⇒ relative minimum at x = a

$f'(x)$ does not change sign at x = a ⇒ No relative maximum or minimum at x = a, then x = a is the point of inflection.

Note: The relative maximum or minimum value (called extreme values) of the function can be found by putting the relative maximum or minimum at x = a in the given function $y = f(x)$

Method of finding maximum or minima

The method is explained with the help of following examples

Example1:

Show that $y = x^5 - 5x^4 + 5x^3 - 1$ has a maximum value when x = 1, a minimum value at x = 3 and neither when x = 0.

Solution:

$$y = x^5 - 5x^4 + 5x^3 - 1$$

$$\frac{dy}{dx} = 5x^4 - 20x^3 + 15x^2$$

$5x^2$

$x^2 = 0$

$x = 0$

Put $\frac{dy}{dx} = 0$ for extreme values.

$$5x^4 - 20x^3 + 15x^2 = 0$$

$$5x^2(x^2 - 4x + 3) = 0$$

$$5x^2(x - 1)(x - 3) = 0$$

$$x = 0, 1, 3$$

For x = 0

When $x < 0, \frac{dy}{dx} = (+ve)\,(-ve)\,(-ve) =$ positive.

$0 < x < 1, \quad \frac{dy}{dx} = (+ve)\,(-ve)\,(-ve) =$ positive.

$\frac{dy}{dx}$ does not change its sign.

Hence x = 0 is a point of inflection.

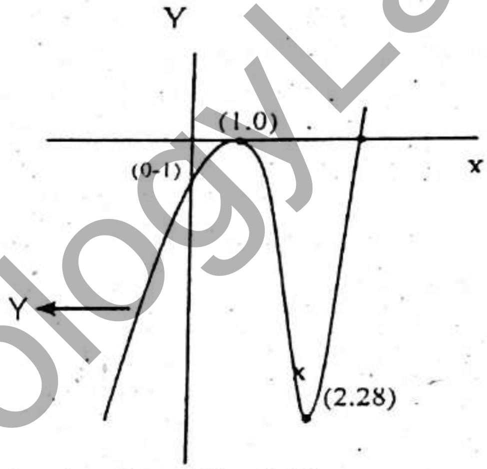


Fig. 4.11

Fig. 4.7

For x = 1

When $0 < x < 1, \frac{dy}{dx} = 5x^2\,(x-1)\,(x-3)$

$= (+ve)\,(-ve)\,(-ve) = +ve$

$1 < x < 3, \quad \frac{dy}{dx} = 5x^2\,(x-1)\,(x-3)$

$= (+ve)\,(+ve)\,(-ve) = -$

$\frac{dy}{dx}$ changes sign from positive o negative

$\therefore \quad x = 1$ is point of maxima.

For x = 3

When $1 < x < 3, \frac{dy}{dx} = 5x^2\,(x-1)\,(x-3)$

$= (+ve)\,(+ve)\,(-ve) = -\,ve$

When $x > 3, \quad \frac{dy}{dx} = 5x^2\,(x-1)\,(x-3)$

$= (+ve)\,(+ve)(+ve) = +ve$

$\frac{dy}{dx}$ changes sign from negative to positive

$\therefore \quad x = 3$ is point of minima

Example 2:

Find maximum and minimum (extreme) values of $x^3(x-4)$

Solution:

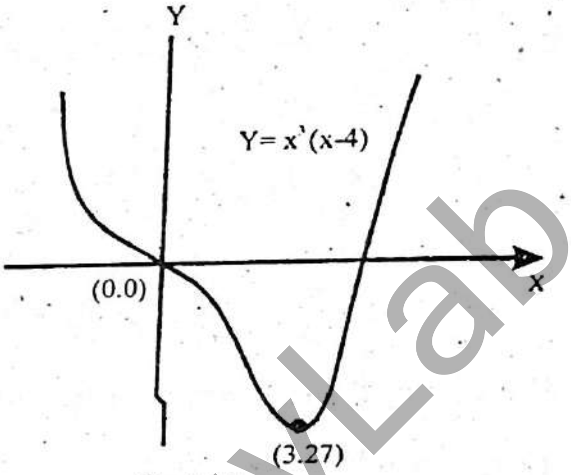


Fig.4.12
Fig. 4.8

$$\text{Let} \quad y = x^3(x-4)$$

$$y = x^4 - 4x^3$$

$$\frac{dy}{dx} = 4x^3 - 12x^2$$

$$= 4x^2(x-3)$$

Put $\frac{dy}{dx} = 0$ for extreme values.

$$\therefore \quad 4x^2(x-3) = 0$$

$$x = 0 \;,\; 3$$

For x = 0

When $x < 0$, $\frac{dy}{dx} = 4x^2(x-3) = 4(+ve)(-ve) = -ve$

When $0 < x < 3$, $\frac{dy}{dx} = 4x^2(x-3) = 4(+ve)(-ve) = -ve$

$\frac{dy}{dx}$ does not change its sign

Hence $x = 0$ is a point of inflection

For x = 3

When $0 < x < 3$, $\frac{dy}{dx} = 4x^2(x-3) = 4(+ve)(-ve) = -ve$

When $x > 3$, $\frac{dy}{dx} = 4x^2(x-3) = 3(+ve)(+ve) = +ve$

$\frac{dy}{dx}$ changes sign from negative to positive, so $x = 3$ gives a minimum point and the minimum value of y is

$$y = x^3(x-4) = (3)^3(3-4) = -27$$

The curve has no maximum point.

4.7 The Second Derivative Test

Consider the function $y = f(x)$ at the point $x = a$ for which $f(x)$ and $f''(x)$ are continuous. The second derivative $y'' = f''(x)$ of y with respect to x is the slope at the point x of the curve representing the derivative $y' = f'(x)$ of the function $f(x)$. In particular $f''(a)$ is the slope of the curve $y' = f'(x)$ at the point $x = a$.

Thus if $f'(x) = 0$ and the second derivative $f''(a)$ of a function $y = f(x)$ is positive. $y' = f'(x)$ is an increasing function of x at $x = a$, in other words slope is increasing as x passes through the pint $x = a$, the $f(x)$ has a minimum value at $x = a$. similarly, if the second derivative $f''(a)$ of a function $y' = f'(x)$ is negative $y' = f'(x)$ is a decreasing function of x at $x = a$; in other words the slope is decreasing as x passes through the point $x = a$, then f(x) has a maximum value at $x = a$, (see **Fig. 4.15**).

If $f''(a) = 0$, then test does not apply.

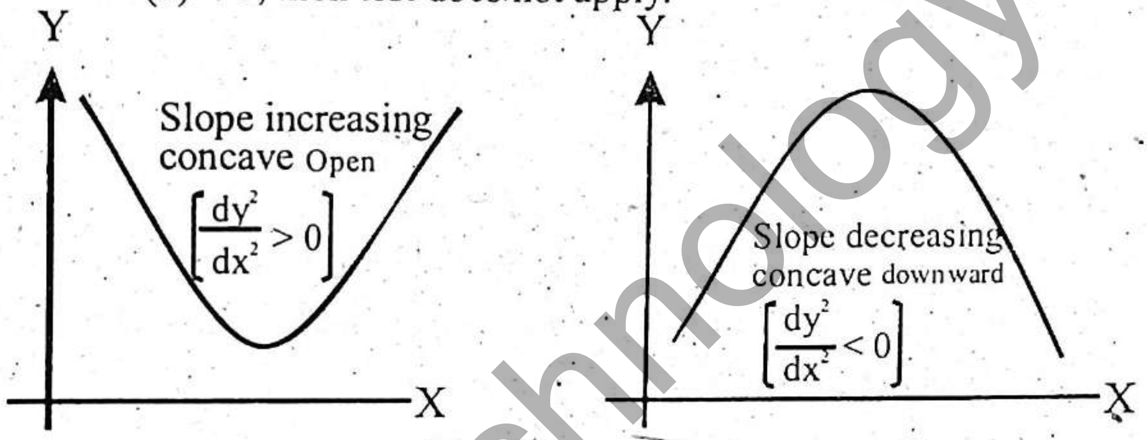

Y
Slope increasing
concave Open
$\left[\frac{dy^2}{dx^2} > 0\right]$
X
Y
Slope decreasing
concave downward
$\left[\frac{dy^2}{dx^2} < 0\right]$
X

Fig. 4.9

4.7.1 Criteria for Finding Maxima and Minima values by 2nd Derivative

1. Find $\frac{dy}{dx}$ and put it equal to zero to find all the values (roots) of x. these are also called critical or turning or stationary points of $y = f(x)$. Let the roots be a, b and c.

2. Find $\frac{d^2y}{dx^2}$ and put one by one the values of x i.e., a, b and c

   (i) If $\frac{d^2y}{dx^2} > 0$ (+ve) at $x = a$, then f(x) is minimum at $x = a$

   (ii) If $\frac{d^2y}{dx^2} < 0$ (–ve) at $x = b$, then f(x) is maximum at $x = b$

   (iii) If $\frac{d^2y}{dx^2} = 0$ at $x = c$, then test does not apply.

To find the maximum and minimum values of f(x), put these values of x in the given equation $y = f(x)$. Also if $f''(x) = 0$ and $f'''(x) \neq 0$ at $x = c$, then the function has point of infection at $x = c$.

4.8 High Order Derivative Test and First Derivative Test

When $f''(a) = 0$ that is, when the second derivative test for maximum or minimum does not apply, the first derivative test is examined for change of sign as x passes through a. There is, however, an alternative procedure, involving derivatives of higher-order, which is more convention in some cases.

$$\text{If} \begin{cases} f'(a) = f''(a) = \ldots = f^{(n-1)}(a) = 0 \\ \text{and} \\ f''(a) \neq 0 \end{cases}$$

$$\text{Then} \begin{cases} \text{for n even} \quad \begin{cases} f''(a) < 0 \Rightarrow \text{relative maximum at } x = a \\ f''(a) > 0 \Rightarrow \text{relative minimum at } x = a \end{cases} \\ \text{for n odd} \quad f(a) \text{ is neither a relative maximum or minimum} \\ \text{i.e., } x = a \text{ is a point of inflection.} \end{cases}$$

4.9 Points of Inflection:

A function $y = f(x)$ is said to have a point of inflection at point where the tangent crosses the curve. At the point of inflection concavity of the curve changes. Since the sign of the second derivative indicates the concavity of a curve, it follows that a change in the sign of the second derivative implies a change in concavity (and thus a point of inflection).

Consider a function f(x) at a value $x = a$ for which f(x) and $f''(x)$ are continuous. If $f''(x)$ changes sign as x passes through the point $x = a$, then the curve representing f(x) has a point of inflection at $x = a$. In this case $f''(a) = 0$.

A point of inflection may also occur at a value $x = a$ for which f(x) continuous but $f''(x)$ is discontinuous. Again, change in the sign of $f''(x)$ as x passes through a is necessary from the existence of a point of inflection at $x = a$.

Thus, in order to determine the points of inflection (if any) of a function $y = f(x)$.

1. Determine the values of x for which $f''(x)$ is either Zero or discontinuous and f(x) is continuous.
2. For each value 'a' determine whether $f''(x)$ change sign as x increases through a.
   If $f''(x)$ changes sign at $x = a \Rightarrow$ Point of infliction at $x = a$

If $f''(x)$ does not change sign at $x = a \Rightarrow$ no point of infliction at $x=a$.

The method is explained with the help of following examples

Example 3:

Find the maximum and minimum values of $(x-1)(x-2)(x-3)$.

Solution:

$$\text{If} \quad y = (x-1)(x-2)(x-3)$$

$$y = x^3 - 6x^2 + 11x - 6$$

$$\frac{dy}{dx} = 3x^2 - 12x + 11$$

$$\text{Put} \quad \frac{dy}{dx} = 0$$

$$\Rightarrow \quad 3x^2 - 12x + 11 = 0$$

$$\Rightarrow \quad x = 2 \pm \frac{1}{\sqrt{3}}$$

$$\frac{d^2y}{dx^2} = 6x - 12$$

$$\text{For} \quad x = 2 + \frac{1}{\sqrt{3}}, \quad \frac{d^2y}{dx^2} = 6\left(2 + \frac{1}{\sqrt{3}}\right) - 12$$

$$= 12 + \frac{6}{\sqrt{3}} - 12$$

$$= \frac{6}{\sqrt{3}} \text{ (positive)}$$

Hence, $x = 2 + \frac{1}{\sqrt{3}}$ gives minimum values, which is

$$y_{\min} = \left(1 + \frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}} - 1\right)$$

$$= \frac{1}{\sqrt{3}}\left(\frac{1}{3} - 1\right) = \frac{-2}{3\sqrt{3}} \quad \text{(from given equation.)}$$

$$\text{For} \quad x = 2 - \frac{1}{\sqrt{3}}, \quad \frac{d^2y}{dx^2} = 6\left(2 - \frac{1}{\sqrt{3}}\right) - 12$$

$$= 12 - \frac{6}{\sqrt{3}} - 12$$

$$= -\frac{6}{\sqrt{3}} \text{ (negative)}$$

Hence $x = 2 - \frac{1}{\sqrt{3}}$ gives maximum value, which is

$$y_{max} = \left(1 - \frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)\left(-1 - \frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3}\right) = \frac{2}{3\sqrt{3}} \text{ (from given equation)}$$

Example 4:

Find the relative maxima and minima of the function.

$$y = \frac{1}{3} x^3 - 2x^2 + 3x + 1$$

Solution:

$$\frac{dy}{dx} = x^2 - 4x + 3$$

For the critical points, put

$$\frac{dy}{dx} = 0$$

So. $x^2 - 4x + 3 = 0$

$(x - 1)(x - 3) = 0$

$x = 1, 3$

$$\frac{d^2y}{dx^2} = 2x - 4$$

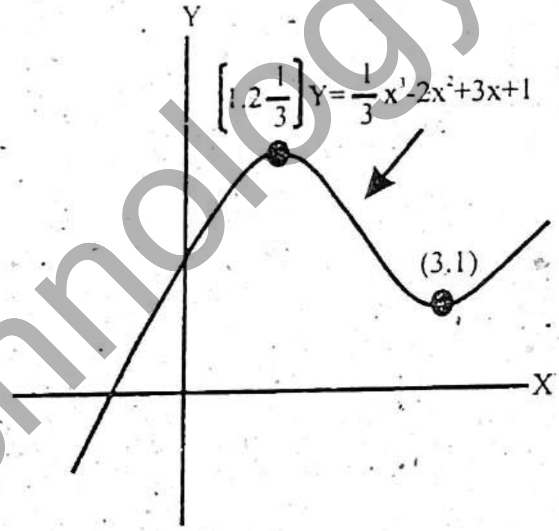


Fig. 4.10

If $x = 1, \frac{d^2y}{dx^2} < 0$ } So maximum at $x = 1$, and maximum value is $y = 2\frac{1}{3}$ And maximum point is $\left(1, 2\frac{1}{3}\right)$

If $x = 3, \frac{d^2y}{dx^2} > 0$ } So minimum at $x = 3$ and minimum value is $y = 1$ And minimum point is $(3, 1)$ Figure 4.16

Example 5:

Find the relative maximum and minimum (if any) of the function $y = x^4$.

Solution:

$$\frac{dy}{dx} = 4x^3$$

for the critical point, put $\frac{dy}{dx} = 0$

So, $4x^3 = 0$

$x = 0$

$$\frac{d^2y}{dx^2} = 12x^2$$

If $x = 0$, $\frac{d^2y}{dx^2} = 0$, so there may or may not be a maximum or minimum at $x = 0$

$$\left.\begin{array}{l} \text{If } x < 0, \frac{d^2y}{dx^2} > 0 \\ \\ \text{If } x > 0, \frac{d^2y}{dx^2} > 0 \end{array}\right\} \text{So no inflection point at } x = 0$$

To check maximum or minimum at $x = 0$, we can apply one of the two tests

A : First Derivative (near – point) test.

$$\left.\begin{array}{l} \text{If } x < 0, \frac{dy}{dx} < 0 \\ \text{If } x > 0, \frac{dy}{dx} > 0 \end{array}\right\} \begin{array}{l} \frac{dy}{dx} \text{ changes from – ve to +ve} \\ \text{So minimum at } x = 0 \end{array}$$

And the minimum values is $y = 0$. The minimum point is (0.0)

B: Higher Order Derivative Test.

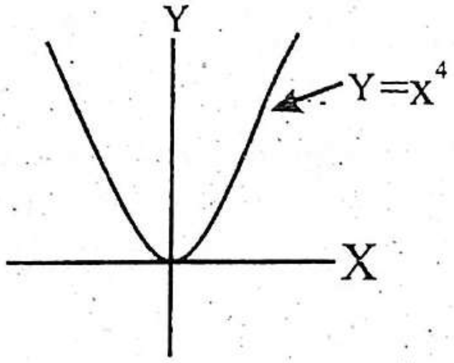


Fig. 4.11

$$\frac{d^3y}{dx^3} = 24x$$

$$\frac{d^4y}{dx^4} = 24$$

If $x = 0$ then $\frac{dy}{dx} = \frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = 0$

and $\frac{d^4y}{dx^4} \neq 0$, n is even

$\frac{d^4y}{dx^4} > 0$, So minimum at $x = 0$

Example 6:

Find the maximum and minimum values of the function.

$y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x$ in the internal $[-3,4]$ or $-3 \le x \le 4$.

Solution:

a. We find the relative maxima and minima.

$$\frac{dy}{dx} = x^2 + x - 2$$

for the critical points, put $\frac{dy}{dx} = 0$

So, $x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2, 1$. Both points lie in the internal $-3 \le x \le 4$.

$$\frac{d^2y}{dx^2} = 2x + 1$$

If $x = -2$, $\frac{d^2y}{dx^2} < 1$, So relative maximum at $x = -2$

And relative maximum value is $y = \frac{10}{3}$

If $x = 1$, $\frac{d^2y}{dx^2} > 0$, So relative minimum at $x = 1$

And relative minimum value is $y = \frac{-7}{6}$

Exercise 4.2

Q.1: Find the critical values(or turning points) for x of the functions:

(i) $5x^2 - 4x + 9$ (ii) $x^2 - 4x - 1$ (iii) $x^2 + 16/x$

(iv) $x^3 - 3x^2 - 24x + 10$ (v) $x^3 + 4x^2 - 3x - 5$

(vi) $2x^4 - x^2$ (vii) $(x+1)(x-1)^2$

Q.2: Find the maximum and minimum (extreme) values of the following functions.

(i) $x^2 - 4x - 6$ (ii) $x^2(x-3)$

(iii) $x^3 - 6x^2 + 9x + 3$ (iv) $\frac{x^3}{3} - \frac{3x^2}{2} + 2x + 5$

(v) $2x^3 - 3x^2 - 36x + 3$ (vi) $(x - 2)^2 (x - 1)$

(vii) $(x - 4)^2 (x - 2)$ (viii) $(x - 2)^3 (x - 3)^2$

Q.3: **Discuss for relative maxima and minima of the following functions:**

(i) $y = x^3 - 3x^2 + 2$ (ii) $y = \frac{2}{3}x^3 - 4x^2 + 6x + 2$

(iii) $y = \frac{1}{4}x^4 - \frac{3}{2}x^2$ (vi) $y = x + \frac{1}{x}$

Q.4: **Find the maximum and minimum values of the function** $\sin x + \cos x$ in the interval $-\pi \le x \le \pi$ or $[-\pi, \pi]$

Q.5. **Find the extreme values of the functions:**

(i) $y = \cos 3x - 3\cos x$ in the interval $(-\pi, \pi)$.
(ii) $y = x + \sin 2x$ in the interval $(-\pi, \pi)$.

Q.6: Show that $x^{1/x}$ has a maximum value for $x = e$

Q.7: Prove that x^x has minimum value at $x = \frac{1}{e}$

Q.8: Show that $\frac{\ln x}{x}$ has a maximum value at $x = e$

Answers 4.2

Q.1: (i) $x = 2/5$ (ii) $x = 2$ (iii) $x = 2$ (iv) $x = -2, 4$

(v) $1/3, -3$ (vi) $x = -1/2, 0, 1/2$ (vii) $x = 1, -1/3$

Q.2: (i) Min= – 10 (ii) Min = – 4; Max= 0
(iii) Max=7 ; Min= 3 (iv) Max =35/6 ; Min =17/3
(v) Max =47 ; Min = – 78 (vi) Min =0 ; Max =4/27
(vii) Max = $\frac{32}{9}$; Min = 0
(viii) 2 is inflection pt; Min= 0 ; Max = 108/3125

Q.3: (i) $x = 0$ is a relative max; max value = 2
$x = 2$ is a relative min; min value = -2

(ii) $x = 1$ is a relative max; max value = 14/3;
$x = 3$ is a relative min; min value = 2 .

(iii) $x = 0$ is a relative max; max value = 0;
$x = \sqrt{3}, -\sqrt{3}$ are relative min; min value = $-9/4$, $9/4$

(iv) $x = -1$ is a relative max; max value = -2

$x = 1$ is a relative Min ; Min value $= 2$

Q.4: Max. at $x = \frac{\pi}{4}$, Max.value $= \sqrt{2}$

Q.5: (i) $x = -\frac{3}{4}\pi\ , 0 , \frac{3}{4}\pi$ are Max points,

Max values $= 2\sqrt{2}\ , -2 , \ 2\sqrt{2}$ (resp)

$x = -\frac{\pi}{4}\ , \frac{\pi}{4}$ are Min. points,

Min. values $= -2\sqrt{2}\ , -2\sqrt{2}$ (resp)

(ii) Max. value $= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$; Min. value $= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

Short questions

Write the short answer of the following:

Note: Notations (symbols): the various notations used for the successive derivations of y or f(x) w.r.t. x are enlisted bellows:

(i) $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^ny}{dx^n}$ (ii) $y_1, y_2, y_3, \ldots\ldots y_n$

(iii) $y', y'', y''', \ldots\ldots y_n$

(iv) $f'(x), f''(x), f'''(x), \ldots f^n(x)$

(v) $f_I(x), f_{II}(x), \ldots\ldots f_n(x)$

(vi) $D_y, D^2y, \ldots D^ny$ $(\therefore D = \frac{d}{dx})$

Q.1: If $y = x^4 - 3x^2 + 4x - 1$, find $\frac{d^2y}{dx^2}$

Q.2: If $y = \ell nx$, find y_2

Q.3: If $y = \cos 3x + \sin 3x$, show that $y_2 + 9y = 0$

Q.4: If $y = Ae^{mx} + Be^{-mx}$, show that $y_2 - m^2y = 0$

Q.5: If $y = \cos(\ell nx)$, show that $x^2y_2 + xy_1 = -y$

Q.6: Find the slope of the tangent to the curve $y = x^2 - 3x + 1$ at the point whose abscissa is 2 i.e., $x = 2$.

Q.7: Find the slope of the tangent to the curve $y = \sin 2x$ at $x = \frac{\pi}{6}$

Q.8: Find slope of tangent to the curve $y = \cos^2x$ at $x = \frac{\pi}{4}$.

Q.9: Find slope of tangent to $x = a\cos\theta$, $y = b\sin\theta$ at $\theta = \frac{\pi}{4}$.

Q.10: Find the slope of tangent to the curve $y = x^3 - 3x + 2$ at (0,2).

Q.11: For the curves $x = t^2 - 1$, $y = t^2 - t$, the tangent is parallel to x – axis, find the value of t.

Q.12: If $x = a\cos\theta$, $y = a\sin\theta$, find $\frac{d^2y}{dx^2}$.

Q.13: Find the turning (or critical) point of the curve $y = \sin 2x$ between 0 and $\frac{\pi}{2}$.

Q.14: Find the turning points of the curve $y = x^2 - 3x + 3$.

Q.15: Find the turning points of the curve $y = 2x^3 - 15x^2 + 36x + 10$.

Q.16: Find the extreme values of the function $y = x^2 - 4x - 6$.

Q.17: Find the derivative of $x = t + e^t$ at $t = 0$ w.r.t. t

Q.18: The radius of a circle is increasing uniformly at the rate of 3 cm per second. Find the rate at which its area is increasing when radius is 10cm.

Q.19: If displacement is $s = \sin 2t$, find its acceleration.

Q.20: If $S = \sin 2t$, find the velocity at $t = \frac{\pi}{6}$

Q.21: Find the acceleration of the moving particle given according to the law $V^2 = 4S - 10$, where S and V have their usual meaning.

Q.22: If $S = \log t$, find the velocity and acceleration at $t = 3$ sec.

Q.23: The distance x meters moved by a particle in t seconds in given by
$x = t^3 + 3t^2 + 4$
Find the velocity and acceleration after 3 seconds.

Answers

Q1. $12x^2 - 6$ **Q2.** $-\frac{1}{x^2}$ **Q6.** 1 **Q7.** 1

Q8. -1 **Q9.** $-b/a$ **Q10.** -3 **Q11.** $t = \frac{1}{2}$

Q12. $\sec^2\theta$ **Q13.** $x = \frac{\pi}{4}$ **Q14.** $x = 2$ **Q15.** $x = 2, 3$

Q16. $y_{min} = -10$ **Q17.** 2 **Q18.** 60π cm²/sec.

Q19. $a = -4 \sin 2t$ **Q20.** $V = 1$ **Q21.** $a = 2$

Q22. $\frac{1}{3}$ m/sec ; $-\frac{1}{9}$ m/sec² **Q23.** 45 m/sec ; 24 m/sec²

Objective Type Questions

Q.1: Encircle the correct one of the given answers in each item.

1. For an increasing function $\frac{dy}{dx}$ is:
   [✓ (a) + ve] (b) - ve (c) zero (d) None of these
2. For a decreasing function $\frac{dy}{dx}$ is :
   (a) + ve [✓ (b) - ve] (c) zero (d) None of these

S.P

3. If $\frac{dy}{dx}$ changes sign from + ve to –ve then it is a point of :
   [✓ (a) maxima] (b) minima
   (c) inflection (d) None of these
4. If $\frac{dy}{dx}$ changes sign from –ve to +ve then it is a point of :
   (a) maxima [✓ (b) minima]
   (c) inflection (d) None of these
5. If $\frac{dy}{dx}$ does not change sign before and after a point where it vanished, then that is a pt. of :
   (a) maxima (b) minima
   [✓ (c) inflection] (d) None of these
6. A function is maximum at a point if $\frac{dy}{dx}$ changes sign from :
   [✓ (a) + ve to –ve] (b) –ve to + ve
   (c) does not change signs (d) zero
7. A function is minimum at a point if $\frac{dy}{dx}$ changes sign from :
   (a) + ve to –ve [✓ (b) –ve to + ve]
   (c) does not change signs (d) zero
8. If 2nd derivative is +ve at a point , then function is :
   (a) maximum [✓ (b) minimum]
   (c) point of inflection (d) None of these
9. If 2nd derivatives is –ve at a point ,then function is :
   [✓ (a) maximum] (b) minimum
   (c) point of inflection (d) None of these
10. A function is maximum at a point if its 2nd derivative is :
    (a) + ve [✓ (b) –ve]
    (c) zero (d) None of these
11. A function is minimum at a point if its 2nd derivative is :

(a) –ve
(b) + ve
(c) zero
(d) None of these

S.P 12. The function $f(x) = x^2$ between $-5 \le x \le -4$ is :

(a) increasing
(b) decreasing
(c) maximum
(d) minimum

Answers

1.	a	2.	b	3.	a	4.	b	5.	c
6.	a	7.	b	8.	b	9	a	10.	b
11.	b	12.	b						

◇◇◇

Chapter 5

Statistics

5.1 Introduction:

The word statistics was derived from the Latin word "Status" or Italian word "statistic". Meaning of both the word is a political state. In the very beginning, it meant for any information useful to the state. Now this word is used in the variant of ways. Statistics play an important role in every field of life. Whether we are at home, at college, in the office statistics is all over there.

Statistics plays an important role in business, banking, insurance, economics, science and engineering. Implementation of statistics spread over all those branches of human knowledge in which a grasp of the significance of large numbers is looked for.

5.2 Definition:

Statistics is a science that deals with the collection presentation analysis and interpretation of numerical facts.

5.3 Data:

The word statistics means numerical data i.e., the numerical facts relating to any filed of study known as data. There are several types of data.

(a) Raw data:

Data which have not arranged in systematic order

Example 1:

Marks of the student in mathematics 49, 36, 35, 15, 18, 48, 75, 73, 42, 32 represents a raw data.

(b) Discrete Data

Data which consists only whole numbers is called discrete data.

Example: Runs record by batsman because there is no fraction in it.

(c) Continuous Data:

Data which consists only continuous variable such as temperature of a particular day of a child etc. there is fraction in continuous data.

5.4 Tabulation:

Systematic arrangement of data into rows and columns is called tabulation.

Example:

Roll No.	English	Math	C. Application	Fail	Pass

5.5 Frequency Distribution:

The arrangement of data according to its size and magnitude is known as frequency distribution or group. Formation of frequency distribution based upon the following facts:

(a) Number of classes to be taken (b) Interval of the class.
(c) Class limits. (d) Frequency in each class.

Group Data:

Data presented in the form of a frequency distribution are also called grouped data.

Class interval:

Suppose we collect data of different students who get marks as shown in the following table.

Classes mark obtained	F = frequency (= No. of students)
0 – 4	3
5 – 9	12
10 – 25	7
26 – 30	10

In the above table we have 0 – 4, 5 – 9, 10 – 25, 26 – 30, as intervals. Where first number in each interval is called lower limit and the second number is called upper limit of the interval i.e.

Lower Limit	Upper Limit
0	4
5	9
10	25
26	30

5.6 Comulative Frequency Distribution:

The total frequency of all values less than the upper class boundary of a given class is called **Comulative Frequency Distribution** or simply **Comulative Distribution. e.g.,**

Weght(kg)	No. of students (f)	Comulative Frequency (cf)
60 – 62	5	5
63 – 65	18	5 + 18 = 23
66 – 68	42	23 + 42 = 65
69 – 71	27	65 + 27 = 92
72 – 74	8	92 + 8 = 100
	$\Sigma f = 100$	

5.7 Class limits and class boundaries:

Consider the class 50 – 54 and 55 – 59 ,. here 50 is the lower limit and 54 is upper limit. Also 49.5 is called the lower class boundary and 54.5 is called the upper class boundary. The lower class boundary is obtained by subtracting 0.5 $\left(\frac{55-54}{2}=\frac{1}{2}=0.5\right)$ from the lower limit and upper boundary is obtained by adding 0.5 in the upper limit.

5.8 Size or length of a class interval:

The difference between the lower and upper class boundaries is called **Size or length of a that interval.** e.g., the length of the class interval 50 – 54 is 54.5 – 49.5 = 5

5.9 Mid point:

The mid point of the class interval 50 – 54 is obtained as $\frac{50+54}{2}=\frac{104}{2}=52$. e.g.,

Class limit	Class boundary	Length of class interval	Mid point
50 – 54	49.5 – 54.5	5	52
55 – 59	54.5 – 59.5	5	67
60 – 64	59.5 – 64.5	5	62
65 – 69	64.5 – 69.5	5	67
70 – 74	69.5 – 74.5	5	72
75 – 79	74.5 – 79.5	5	77
80 – 84	79.5 – 84.5	5	82
85 – 89	84.5 – 89.5	5	87
90 – 94	89.5 – 94.5	5	92
95 – 99	94.5 – 99.5	5	97

5.10 Averages, Mean, Median and Mode:

A value around the whole data is scattered, known as measures of central tendency or averages. The following are the important types of averages.

5.11 Types of Averages:

1. Arithmetic mean, arithmetic average or simply mean
(2) Median 3. Mode

5.11.1 The Mean

The arithmetic mean or simply the mean in defined as the value obtain by dividing the sum of the values by their number. This is used in finding the batting average in crickets, in finding average number of goals scored in football and so on.

(a) Mean for Ungrouped Data:

The A.M. of the values $x_1, x_2, x_3, \ldots\ldots\ldots\ldots x_n$, is denoted by $\overline{X}$ and is given by

$$A.M. = \overline{X} = \frac{x_1 + x_2 + \ldots\ldots\ldots\ldots\ldots + x_n}{n}$$

$$= \frac{\Sigma x}{n}$$

Where n = Total number of values

Example 2: Find the mean of the numbers 5, 8, 10, 12 and 7

$$A.M. = \overline{X} = \frac{5+8+10+12+7}{5} = \frac{42}{5} = 8.4$$

(b) Mean for Grouped Data:

If the number of values in the data are very large, then it is divided into classes with frequency distribution. Then for group data following formula is used.

$$A.M. = \overline{X} = \frac{f_1 x_1 + f_2 x_2 \ldots\ldots\ldots\ldots\ldots + f_n x_n}{n} = \frac{\Sigma fx}{n}$$

Here $n = f_1 + f_2 + f_3 + \ldots\ldots\ldots\ldots + f_n = \Sigma f$

Example 3: Find the mean for the following distribution showing marks obtained by 50 students in English.

Marks	20 – 24	25 – 29	30 – 34	35 – 39	40 – 44	45 – 49	50 - 54
Frequency	1	4	8	11	15	9	2

Solution: Construct the following table:

Mark	Frequency (f)	Class marks (x)	(f x)
20 – 24	1	22	22
25 – 29	4	27	108
30 – 34	8	32	256
35 – 39	11	37	407
40 – 44	15	42	630
45 – 49	9	47	432
50 – 54	2	52	104
Total:	$n = \Sigma f = 50$		$N = \Sigma fx = 1950$

Here, x = Average of class marks in each group (or interval)

$$n = 50 \text{ and } \Sigma fx = 1950 \quad \Rightarrow \quad AM = \overline{X} = \frac{\Sigma fx}{n} = \frac{1950}{50} = 39 \text{ marks}$$

Short Method for Finding Mean

(c) Mean for Ungrouped Data:

When values of x and f are large then short method is very useful. In this method any provisional mean is guessed and deviations are considered from the provisional mean P.M. Then Deviations are added and this sum is divided by the number of items is added to it giving the required mean.

$$\therefore \quad \text{A.M.} = \overline{X} = \text{P.M.} + \frac{\Sigma D}{n}$$

When P.M. = provisional mean
D = deviation of value from p.m.

Example 4: Find the mean of following values using short method 184, 191, 172, 168, 187, 189, 196, 189, 193 and 195.

Solution: Let assumed mean = 180

Now take deviations from 180

$\therefore$ D = 4, 11, –8, –12, 7, 9, 16, 9, 13, 15

$\Sigma D = 64$

$$\therefore \quad \text{A.M.} = \overline{X} = \text{P.M.} + \frac{\Sigma D}{n}$$

$$= 180 + \frac{64}{10} = 186.4$$

(d) Mean for Group Data:

The short formula to find the mean for grouped data is

$$\text{A.M.} = \overline{X} = \text{P.M.} + \frac{\Sigma f D}{n}$$

Example 5: Find the mean weight of 120 students of a college for the distribution of weights.

Solution: The computation of means is calculated with the help of following table:

Class marks x	Frequency f	Deviation D= x – 144.5	fD
114.5	1	– 30	–30
124.5	4	–20	–80
134.5	17	–10	–170
144.5	28	0	0
154.5	25	10	250
164.5	18	20	360
174.5	13	30	390
184.5	6	40	240
194.5	5	50	250
204.5	2	60	120
214.5	1	70	70
Total:	**N = Σf = 120**		**N = Σ fd = 1680 – 280 = 1400**

$$\therefore \quad \text{A.M.} = \overline{X} = \text{P.M.} + \frac{\Sigma f D}{n}$$

$$= 144.5 + \frac{1400}{120} = 144.5 + 11.67 = 156.17 \text{ Pounds}$$

Example 6: Frequency distribution given below in the table of production of engineering products.

Cost of Production	**f**
0 – 4	5
4 – 8	7
8 – 12	9
12 – 16	6
16 – 20	3

Calculate A.M. for the given data

Solution: Let P.M. = 10

Cost of Production	F	X	Fx	D=x−10	f.D	$D'=D/4$	fD′
0 – 4	5	2	10	−8	−40	−2	−10
4 – 8	7	6	42	−4	−28	−1	−7
8 – 12	9	10	90	0	0	0	0
12 – 16	6	14	84	4	24	1	6
16 – 20	3	18	54	8	24	2	6
	30		280		−20		−5

Here $i = 4$

$$\text{A.M.} = \text{P.M.} + \frac{\Sigma f D'}{\Sigma f} \times i$$

$$= 10 + \frac{(-5)}{30} \times 4 = 10 + \frac{-20}{30}$$

$$= 10 - \frac{20}{30} = 9.333$$

MEDIAN

5.12 Definition:

Median is that value which divides the data into two equal parts. When the data are arranged in order , it is usually denoted by $\tilde{x}$. If there is an even number of value, we take the mean of the two middle values. Thus we calculate the median as

(i) when n is odd $\quad \text{Median} = \left(\frac{n+1}{2}\right)\text{th value}$

(ii) when n is even $\quad \text{Median} = \frac{1}{2}\left[\frac{n}{2}\text{th} + \left(\frac{n}{2}+1\right)\text{th value}\right]$

5.12.1 Median For Ungrouped Data:

Example 7 (i) Find the median of 4, 3, 5, 2, 11,
Write the values in ascending order 2, 3, 4, 5, 11
Here $n = 5$

$$\text{Median} = \left(\frac{n+1}{2}\right)\text{th value}$$
$$= \left(\frac{5+1}{2}\right)\text{th} = 3\text{th value}$$

i.e. The median is the third value which is = 4

Example 7 (ii) Find the median of 4, 3, 5, 2, 6, 11
Write the value in ascending order 2, 3, 4, 5, 6, 11
Here $n = 6$

$$\text{Median} = \frac{1}{2}\left[\frac{n}{2}\text{th} + \left(\frac{n}{2}+1\right)\text{th value}\right]$$
$$= \frac{1}{2}\left[\frac{6}{2}\text{th} + \left(\frac{6}{2}+1\right)\text{th value}\right]$$
$$= \frac{1}{2}(3^{th} + 4^{th})$$
$$= \frac{1}{2}(4+5) = \frac{9}{2}$$

∴ The median is $4\frac{1}{2}$

5.12.2 Median for Grouped Data:

Median for the grouped data is given by

$$\text{Median} = \ell + \frac{h}{f}\left(\frac{n}{2} - c\right)$$

When ℓ = lower limit of the relative class interval.
h = class interval.
n = Total number of frequency.
C = Cumulative frequency of the group preceeding the median group
f = Frequency of the median group

Example 8: Calculate the median from the following table.

Class	f	cf
0 – 5	3	3
5 – 10	7	10
10 – 15	14	24
15 – 20	16	40
20 – 25	**22**	**62**
25 – 30	18	80
30 – 35	10	90
35 – 40	6	96
40 – 50	3	99
45 – 50	1	100
	$\Sigma f = n = 100$	

The class which corresponds to $\frac{n}{2}$ th value is called a median class.

That is median class = $\frac{n}{2}$ th item = $\left(\frac{100}{2}\right)$ th item

= 50^{th} item

$\therefore$ 50 lies in 62 class

Here C.f of preceeding the median class is C = 40

$$\text{Median} = l + \frac{h}{f}\left(\frac{n}{2} - c\right)$$

Here ℓ = 20 lower limit of median class

h = 5 = class interval

f = 22 = frequency of the median class

c = 40 = cumulative frequency of the group presenting in the median class

n = 100 = the median class frequency

$$\therefore \text{Median} = 20 + \frac{5}{22}\left(\frac{100}{2} - 40\right)$$

$$= 20 + \frac{50}{22} = \frac{440 + 50}{22.} = \frac{490}{22} = 22.2727$$

Example 9: Calculate the median from the following table.

Solution :

Class	f
65– 84	7
85 – 104	6
105 – 124	8
125 – 144	2
145 – 164	2
165 – 184	2
185 – 204	3

$$\text{Median} = \tilde{x} = \ell + \frac{h}{f}\left(\frac{n}{2} - c\right)$$

The necessary calculation are as given

Class intervals	Class boundaries	Frequency (f)	Cumulative (c f)
65 – 84	64.5 – 84.5	7	7
85 – 104	84.5 – 104.5	6	13
105 – 124	**104.5 – 124.5**	**8**	**21**
125 – 144	124.5 – 144.5	2	23
145 – 164	144.5 – 164.5	2	25
165 – 184	164.5 – 184.5	2	27
185 - 204	184.5 – 204.5	3	30
		$\Sigma f = n = 30$	

Now, $\frac{n}{2}$ th value = $\frac{30}{2}$ = 15 th value

The class which corresponds to 15 th value is 104.5 – 124.5

Thus $l = 104.5$, $h = 104.5 - 124.5 = 20$, $f = 8$, $c = 13$

$$\text{Median} = \ell + \frac{h}{f}\left(\frac{n}{2} - c\right)$$

$$= 104.5 + \frac{20}{8}(15 - 13) = 104.5 + \frac{5}{2}(2)$$

$$= 109.5$$

MODE

5.13 Definition:

A value which occurs maximum number of times in the data is called a Mode.

Note : A set of data may have more than one mode or no mode at all.

5.13.1 Mode for Ungrouped Data:

Find the mode of 2, 3, 4, 4, 4, 6, 8, 9, 9, 9, 9, and 10

The 9 occurs most frequently and is the mode

Example 10: Find the mode of 1, 2, 5, 6, 12 and 13.

Above data has no mode.

5.13.2 Mode for grouped Data:

There are three different formulas to calculate mode

1. $$\text{Mode} = \ell + \frac{f_m - f_1}{(f_m - f_1) + (f_m - f_2)} \times h$$

2. $\text{Mode} = \ell + \dfrac{f_2}{f_1 + f_2}$

3. Mode = 3 median – 2 A.M.

Where
ℓ = lower limit of the model class
f_m = Frequency of the model class (highest frequency)
f_1 = Frequency of the class preceding the model class
f_2 = Frequency of the class following the model class.
h = Class interval.

Example 11: Calculate mode from the following data.

Marks	No. Students
0 – 10	2
10 – 20	7
20 – 30	11
30 – 40	6
40 – 50	4

Solution:

Marks	No. Students	
0 – 10	2	
10 – 20	7	$\rightarrow f_1$
20 – 30	11	$\rightarrow f_m$
30 – 40	6	$\rightarrow f_2$
40 – 50	4	

Here
$\ell = 20$ $f_m = 12$
$f_1 = 7$ $f_2 = 6$
$h = 10$

Formula 1: Mode = $= \ell + \dfrac{f_m - f_1}{(f_m - f_1) + (f_m - f_2)} \times h$

$$= 20 + \frac{(11 - 7)}{(11 - 7) + (11 - 6)} \times 10$$

$$= 20 + \frac{4}{4 + 5} \times 10$$

$$= 20 + \frac{40}{9} = 20 + 4.444 = 24.44$$

Formula 2: Mode $= \ell + \dfrac{f_2}{f_1 + f_2} \times h$

$$= 20 + \frac{6}{7 + 6} \times 10$$

$$= 20 + 4.615$$

$$= 24.615$$

Formula 3: Mode = 3 Median – 2 A.M.

Construct the following table

Class	f	x	fx	cf
0 – 10	2	5	10	2
10 – 20	7	15	105	9
20 – 30	**11**	**25**	**275**	**20**
30 – 40	6	35	210	26
40 – 50	4	45	180	30
	Σ f = 30		**Σ fx = 780**	

Here location for median class = $\frac{n}{2}$ th = $\frac{30}{2}$ th = 15 th value

$\ell = 20$, $f = 11$, $h = 30 - 20 = 10$, $c = 9$, $n = 30$

$$\text{A.M.} = \frac{\Sigma fx}{\Sigma f} = \frac{780}{30} = 26$$

$$\text{Median} = \ell + \frac{h}{f}\left(\frac{n}{2} - c\right)$$

$$= 20 + \frac{10}{11}(15 - 9)$$

$$= 20 + \frac{10 \times 6}{11} = 20 + 5.455 \qquad = 25.455$$

$$\therefore \quad \text{Mode} = 3\ \text{Median} - 2\ \text{A.M.}$$

$$= 3 \times 25.455 - 2 \times 26$$

$$= 76.365 - 52.000 \qquad = 24.365$$

Exercise 5.1

Q1. Find the mean median and mode of the following scores.

(i) 2, 3, 3, 3, 5 (ii) 4, 0, 2, 9, 0.
(iii) 0, 1, 4, 5, 9, 9 (iv) 5, 0, 1, 2, 0, 7

Q2. Find the mean for each of the following sets of data

(i) 60, 62, 68, 72, 76, 86, 85, 83 and 85
(ii) 10, 17, 29, 95, 95, 100, 100, 175, 250 and 750

Q3. Calculate A.M. Median from the following data:

Marks 15, 17, 18, 19, 20, 25, 26, 27, 28, 29, 30, 31
Boys: 30, 34, 9, 38, 15, 50, 52, 81, 56, 58, 15, 62

Q4. The following table gives the marks obtained by a batch of 5 students in an examination in English, Mathematics and Computer Applications.

Roll No.	English	Mathematics	Computer Application
1	41	46	50
2	35	50	52
3	38	39	41
4	34	50	46
5	30	38	39

In which subject the level of knowledge is highest.

Q5. Find the arithmetic mean (A.M.) from the following frequency distribution.

Marks	Frequency
11 – 25	6
25 – 40	20
40 – 55	44
55 – 70	26
70 – 85	3
85 – 100	1

Q6. Compute mean and median from the data given above.

Class Interval	Frequency
0 – 5	4
5 – 10	6
10 – 15	10
15 – 20	16
20 – 25	12
25 – 30	8
30 – 35	4

Q7. Calculate the mode of the following data:

Weights	4	5	6	7	8	9	10	11	12	13
Frequency	2	5	8	9	12	14	14	10	11	13

Q8. Class Interval 1 – 3, 4 – 6, 7 – 9, 10 – 12, 13 – 15 and 16 – 18
Frequency 12 5 20 22 14 and 17
Calculate A.M., Median and Mode.

Q9.

Weekly wages	No of Workers
0 – 4	6
4 – 8	15
8 – 12	22
12 – 16	28
16 – 20	45
20 – 24	25
24 – 28	13
28 – 32	6

Calculate A.M., Median and Mode

Q10. For the following frequency table calculate the arithmetic mean, median and mode.

Height in Cm.	59	58	57	56	55	54	53	52	51	50	49	48	47
No. of Boys	1	3	7	8	25	30	55	50	40	38	30	9	4

Answers

1. (i) 3.2, 3, 3 (ii) 3, 2, 0 (iii) $4\frac{2}{3}, 4\frac{1}{2}, 9$ (iv) $2\frac{1}{2}, 1\frac{1}{2}, 0$
2. (i) A.M =752 (ii) A.M =162.1
3. A.M. = 25.246 Median = 27 approx,
4. Computer Application = 228
5. A.M = 47.95 6. A.M =18, Median = 18 7. Mode =9
8. A.M = 9.86, Median = 9.41, Mode = 10.6 approx
9. A.M. = 16.2 , Median = 16.8 , Mode = 17.838
10. A.M. = 52.07, Median = 52.08 (approx) Mode = 52.67 (approx)

Measures of Dispersion

5.14 Introduction:

Measures of central tendency inform us about the value around which the values of individual items vary. Now we attempted to find measures of dispersion. The measures used for this purpose are called measures of dispersion or measure of variation. There are different ways of measuring dispersion but following are commonly used.

1. Mean Deviation 2. Standard Deviation

5.14.1 Standard Deviation

The positive square root of mean of the squares of all deviations taken from the arithmetic mean.

The standard deviation of a set of n value $x_1, x_2 \ldots\ldots\ldots x_3$, which is denoted by σ (read as sigma) is given as.

(1) For ungrouped data

$$\sigma = \sqrt{\frac{\Sigma (X - \overline{X})^2}{n}} \quad \text{or} \quad \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$$

(2) For grouped data

$$\sigma = \sqrt{\frac{\sum f\,(x-\bar{x})^2}{\sum f}} \quad \text{or} \quad \sqrt{\frac{\sum f x^2}{\sum f} - \left(\frac{\sum f x}{\sum f}\right)^2}$$

Example 3: Find the standard deviation from the values 2, 3, 6, 8, 11.

Solution: Here, $\overline{X} = \frac{2+3+6+8+11}{5} = \frac{30}{5} = 6$

$$\sigma = \sqrt{\frac{\sum (X-\overline{X})^2}{n}}$$

$$= \sqrt{\frac{(2-6)^2+(3-6)^2+(6-6)^2\,(8-6)^2+(11-6)^2}{5}}$$

$$\text{S.D} = \sigma = \sqrt{\frac{16+,9+0+4+25}{5}} = \sqrt{\frac{54}{5}} = \sqrt{10.8} = 3.28$$

Example 4: Find in standard deviation for the following data.

Marks	10 – 20	20– 30	30– 40	40– 50	50– 60	60– 70	70– 80
f	6	12	20	24	10	4	4

Solution:

Marks	f	X	fx	$\lvert X-\overline{X}\rvert$	$(X-\overline{X})^2$	$f(x-\overline{x})^2$
10 – 20	6	15	90	26	676	4056
20 – 30	12	25	300	16	256	3072
30 – 40	20	35	700	6	36	720
40 – 50	24	45	1080	4	16	284
50 – 60	10	55	550	14	196	1960
60 – 70	4	65	260	24	576	2304
70 – 80	4	75	300	34	1156	4624
	$n = \sum f = 80$		3280			17120

$$\text{A.M.} = \frac{\sum fx}{\sum f} = \frac{3280}{80} = 41$$

$$\text{S.D.} = \sigma = \sqrt{\frac{\sum f\,(x-\bar{x})^2}{\sum f}} = \sqrt{\frac{17120}{80}} = \sqrt{\frac{1712}{8}}$$

$$= \sqrt{214} = 14.629$$

Exercise 5.2

1. Calculate the S.D from the Mean for the following data, 2, 6, 9, 12, 8, 13, 5, 6, 23, 16
2. Calculate the S.D from the Mean for the following data 68, 49, 32, 21, 54, 38, 59, 66, 41.
3. Find the median and the standard deviation from the mean for the data given:

Daily wages	6	8	10	12	14	16	18	20	22
No of Servant	5	10	18	20	22	14	7	3	1

4. Find the standard deviation from the mean of following data.

Marks	**No. of Student**
0 – 2	2
2 – 4	10
4 – 6	26
6 – 8	32
8 – 10	8
10 – 12	2

5. Find the S.D. from following data.
5, 9, 13, 17, 19, 25, 30, 32, 33, 35, 40, 41
6. Calculate the S.D from the following data

Size Item	2	3	4	5	6	7	8	9	10	11
Frequency	9	6	2	2	2	4	3	3	2	3

7. Determine standard deviation from the following frequency distribution.

Group	20-- 24	25– 29	30– 24	35– 39	40– 44	45– 49	50 - 54
F	1	4	8	11	15	9	2

8. Calculate standard deviation from the following data.

Weekly wages	40–45	44–50	50–55	55–60	60–65	65–70	70–75	75–80
No. of persons	10	12	17	9	14	24	18	9

9. Calculate the standard deviation from the following data
Group : 20 – 24 , 25 – 29 , 30 – 34 , 35 – 39 , 40 – 44 , 45 – 49 , 50 – 54
Frequency : 01 , 04 , 08 , 11 , 15 , 09 , 02
10. From the following data calculate the S.D
Height in inches : 60 – 62 , 63 – 65 , 66 – 68 , 69 – 71 , 72 – 74
No. of students : 5 , 18 , 42 , 27 , 8

Answers

1. S.D = 5.86 2. S.D = 14.96 3. Median = 12 , S.D= 5.54
4. S.D = 2 5. S.D. = 11.61 6. S.D = 3.11
7. S.D. = 6.85 8. S.D. = 10.4 9. S.D= 6.85 10. S.D= 2.93

Solved Short Questions

Write the short answers of the following :

Q.1: Define statistics.

Q.2: What is primary data.

Q.3: Define Secondary data.

Q.4: Define the terms "Population" and a "Sample".

Q.5: What is a frequency distribution?

Q.6: What is the class interval?

Q.7: Which of the following variables are discrete and which are continuous.

(i) No. of persons in a family. (ii) No. of books in a rack.
(iii) Age of a person. (iv) Price of rice.
(v) Height of a child.

Q.8: If Mode = 15 , median = 12 , find Mean

Q.9: Define arithmetic mean.

Q.10: The arithmetic mean of 7 values is 6, find the sum of values.

Q.11: Define median and mode.

Q.12: Find median and mode of the following data.

X :	2	4	6	8	10
f:	13	20	22	18	10

Q.13: Calculate the median for 88.03 , 94.50 , 94.90 , 95.05 , 84.50

Q.14: Write formula to find median for grouped frequency distribution?

Q.15: Define standard deviation.

Q.16: How can we calculate the median for odd data.

Q.17: Find S.D. of the values, 2, 4, 6, 8, 10.

Q.18: Write formula to find mode of a grouped frequency distribution.

Q.19: How can we calculate the median for even data

Q.20: Find median and mode of the values 2, -1, 7, 0, -4, 2, 5.

Q.21: Find mean of the data.

x :	1	3	5	7	9
f :	2	7	11	6	4

Q.22: Find median of the grouped data.

Group :	0 – 4	5 – 9	10 – 14	15 – 19
f :	2	9	11	8

Answers

Q.1: Statistics is the science of estimates and probabilities.

Q.2: It is most original data and has not undergone any statistical treatment. In other words, it is first hand information.

Q.3: It is that Data which has gone through statistical treatment at least once.

Q.5: A tabular arrangement of data, in which the values of a variable are grouped along with their respective frequencies.

Q.6: The difference between upper and lower class boundaries is the class interval of that class.

Q.7: Discrete: (i) (ii) (iv)
Continuous: (iii) (v)

Q.8: **10.5**

Q.9: Let $x_1, x_2, x_3, ,,,, x_n$ be n values of a variable x, then their arithmetic mean is denoted and defined as $\bar{x} = \frac{\Sigma x_i}{n}$; $i = 1, 2, 3, \ldots, n$

Q.10: 42

Q.11: **Median:** the value which divides an arrangement data into two equal parts is known as median.

Mode: It is defined as the most frequent value in a data.

Q.12: **Median = 22 , Mode = 54.4**

Q.13: **Median = 94.50**

Q.14: Median $= \ell + \frac{h}{f}\left(\frac{n}{2} - c\right)$

Q.15: Let $x_1, x_2, x_3, \ldots\ldots, x_n$ be n values of a variable x, then their standard deviation is denoted and defined as

$$S.D. = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n}} \quad ; \quad i = 1, 2, 3, .., n$$

Q.16: Median $= \left(\frac{n+1}{2}\right)$th value

Q.17: S.D = 2.84

Q.18: Mode $= \ell + \frac{f_m - f_1}{(f_m - f_1) + (f_m - f_2)} \times h$

Q.19: Median = $\frac{1}{2}\left[\frac{n}{2}\text{th} + \left(\frac{n}{2}+1\right)\text{th}\right]$ value

Q.20: median = 2 , mode = 2

Q.21: Mean = 5.2

Q.22: Median = 11.3

Find the median of
the values:

2, -1, 7, 0, -4, 2, 5

Objective Type Questions

Q.1: Encircle the correct one of the given answers in each item.

S.P 1--- First hand information is called
(a) Primary data
(b) Secondary data
(b) Raw data
(d) Continuous data

2--- A set of data is called
(a) Continuous data
(b) Discontinuous data
[✓ (c) Population]
(d) Sample

S.P 3--- Subset of a population is called
(a) Raw data
(b) Secondary data
(c) Population
[✓ (d) Sample]

4--- A table showing the cumulative frequency is called
[✓ (a) Cumulative frequency distribution]
(b) Sample frequency
(c) Normal distribution
(d) Binomial distribution

5--- The difference between the lower and upper class boundaries is called
(a) Common difference
[✓ (b) Size or length of interval]
(c) Difference operator
(d) Size of the table

6--- Sum of the lower and upper class limits / boundaries divided by two is called
(a) End point
(b) Important point
[✓ (c) Mid point]
(d) Significant point

7--- A quantities whose value varies is called
(a) Constant
(b) Parameter
(c) Variable
[✓ (d) Function]

8--- A quantities whose value remains fixed is called
[✓ (a) Constant]
(b) Variable
(c) Parameter
(d) Function

9--- Mean , Median and Mode are the types of
[✓ (a) Average]
(b) Function
(c) Variable
(d) Constant

10--- The sum of all variables divided by their number is called
(a) Median
[✓ (b) Arithmetic Mean]
(c) Mode
(d) Geometric Mean

11--- When n is odd, $\left(\frac{n+1}{2}\right)$th value is called
(a) Mean
(b) Mode
[✓ (c) Median]
(d) Harmonic Mean

12--- The formula $\tilde{x} = \ell + \frac{h}{f}\left(\frac{n}{2} - c\right)$ is used for finding
[✓ (a) Median for grouped data]
(b) Arithmetic Mean

(c) Geometric Mean (d) Mode

13--- In the formula $\tilde{x} = \ell + \frac{h}{f}\left(\frac{n}{2} - c\right)$, ℓ denotes

(a) class interval
[✓ (b)] lower class boundary of the Median class
(c) frequency of the Median class
(d) commulative frequency preceeding to the Median class

14--- In the formula $\tilde{x} = \ell + \frac{h}{f}\left(\frac{n}{2} - c\right)$, h denotes

(a) Total number of frequency
(b) Lower class boundary of the Median class
[✓ (c)] Class interval
(d) Frequency of the Median class

15--- In the formula $\tilde{x} = \ell + \frac{h}{f}\left(\frac{n}{2} - c\right)$, c denotes

(a) class interval
[✓ (b)] commulative frequency preceeding to the Median class
(c) number of frequency
(d) frequency of the Median class.

16--- A value which occurs maximum number of times in the data is called
(a) Mean (b) Geometric Mean
(c) Harmonic Mean [✓ (d)] Mode

17--- The formula $\ell + \frac{f_m - f_1}{(f_m - f_1)(f_m - f_2)} \times h$ is used for finding

[✓ (a)] Mode for grouped data (b) Median
(c) Mean (d) Geometric Mean

18--- Mode can be calculated by the formula
(a) Mode = 3 Median – 4 Mean
(b) Mode = 4 Median – 3 Mean
[✓ (c)] Mode = 3 Median – 2 Mean
(d) Mode = 2 Median – 3 Mean

19--- The formula for calculating Standard Deviation is

(a) $\sigma = \sqrt{\frac{\Sigma(x - \bar{x})}{n}}$ (b) $\sigma = \sqrt{\frac{\Sigma\{x^2 - x\}}{n}}$

(c) $\sigma = \frac{\Sigma x}{n}$ [✓ (d)] $\sigma = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n}}$

20--- The formula for calculating Standard Deviation for grouped data is

[✓ (a)] $\sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}}$ (b) $\sqrt{\frac{\sum f\,(x-\bar{x})}{\sum f}}$

(c) $\sqrt{\frac{\sum f\,(x^2-x)}{\sum f}}$ (d) $\sqrt{\frac{\sum f\,(x-\bar{x}^2)}{\sum f}}$

Answers

Q1.

1.	a	**2.**	c	**3.**	d	**4.**	a	**5.**	b
6.	c	**7.**	d	**8.**	a	9.	a	**10.**	b
11.	c	**12.**	a	**13.**	b	**14.**	c	**15.**	b
16.	d	**17.**	a	**18.**	c	**19.**	d	**20.**	a

Chapter 6

Probability

6.1 Introduction:

Probability is a familiar concept in our daily life. Probability is often referred to as chances" or " Odds". When the sex of an expected child, for example, we say the chances are about "fifty fifty", that it will be a boy. We may predict the out come of some event by saying that chances of happening are 2 to 1 or that 10 to 1. These are probability statements.

The first step in considering probability formally is to establish a numerical representation of probability. If there is absolute certainty of happening of some outcome, the probability is 1. But if some outcome is absolutely not to occur then probability is zero. Any intermediate probability is represented by some value between 0 and 1.

Probability theory has an important role in making important decisions in uncertain situations. It has a wide range of applications in many fields. Before defining it, some very important terms are being given, which will be used in solving different problems and proving theorem of probability in this chapter.

6.2 Experiment and Random Experiment:

An experiment is a process of getting an observation and if, under similar conditions an experiment produces different results, even if it is repeated many times it is called a random experiment. We cannot predict next out come.

Tossing of a coin or rolling of die are examples of random experiment.

6.3 Sample Space:

Set of all the possible outcomes of an experiment is called sample space. Each out come is called a sample point. Some examples of sample space are given below.

1. Tossing of a coin {H,T}
2. Tossing of two coins {HH, HT, TH, TT}
3. Rolling of die {1, 2, 3, 4, 5, 6}
4. Rolling of two dice { (1,1), (1,2), (1,3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), 2, 5), (2, 6), (3 ,1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),, (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6),}

Sample space is usually denoted by S.

6.4 Event:

It is a subset of a sample space. For example considering the sample space of two coins tossed {HH, HT, TH, TT} A = {HH, HT} and B = {HH, TH} are event. An event may be simple or compound. If there is only one sample point in the event, it is simple otherwise a compound event.

Simple events from the above sample space are {HH}, (HT}, {TH} and {TT}.

Events are of many types and play an important role in the theory of probability. Some of its types are given below.

6.4.1 Mutually Exclusive Events:

Event A and B are mutually exclusive, if both of them cannot occur at the same time. In tossing a coin head and tail cannot occur at the same time. There may be some events which can occur at the same time, even then they are mutually exclusive.

For example, from a well shuffled pack of 52 cards, a randomly drawn card is spade and a king as well. So it is both a spade and a king.

6.4.2 Equally Likely Events:

If the chances of happening of every outcome are same, the events are called equally likely. For example when a coin is tossed, the chances of a head and a tail are same.

6.4.3 Exhaustive Events:

Events are exhaustive, if they constitute the entire sample space. When a coin is tossed, events are {H} and {T}. there union is a sample space.

6.5 Definition of Probability:

If we have a mutually exclusive, equally likely and exhaustive sample points in a sample space and m of them are favourable for happening in event A then.

$$P(A) = \frac{\text{number of favourable points}}{\text{total number of sample points}}$$

If for example we throw a die, sample space is {1, 2, 3, 4, 5, 6} and favourable points are 2 and 6 in event A then

$$P(A) = \frac{\text{favourable points}}{\text{sample points}} = \frac{2}{6} = \frac{1}{3}$$

Example 1:

From a well shuffled pack of 52 cards, a card is drawn at random. Find the probability that it is a (1) king (2) card of spade (3) black card (4) a pictured card.

Solution:

Total number of mutually exclusive, equally likely and exhaustive cases = 52

1. Number of favourable cases to the happening of a king = 4 ($\because$ there are 4 kings in the pack of 52)

$$\text{Probability of king} = \frac{4}{52} = \frac{1}{13}$$

2. Number of favourable cases to the happening of a spade card = 13 ($\because$ there are 13 spades in the pack of 52).

$$\text{Probability of spade} = \frac{13}{52} = \frac{1}{4}$$

3. Number of favourbale cases to the happening of a black card = 26 ($\because$ there are 26 black cards in the pack of 52).

$$\text{Probability of a black card} = \frac{26}{52} = \frac{1}{2}$$

4. Number of favourable cases to the happening of a pictured card = 12 ($\because$ there are 12 pictured cards in the pack of 52).

$$\text{Probability of a pictured card} = \frac{12}{52} = \frac{3}{13}$$

Example 2:

A slip is picked out of 10 slips numbered 1, 2, 3,.......10. what is the probability to get the

1. Slip number divisible by 3 2. Slip number not divisible by 3

Solution:

Sample space {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Let A = Event containing slip number divisible by 3 = {3, 6, 9}

$$P(A) = \frac{3}{10}$$

B = Event containing slip number not divisible by 3 = {1, 2, 4, 5, 7, 8, 10}

$$P(A) = \frac{7}{10}$$

Example 3:

If a pair of dice is rolled. What is the probability that

1. A sum of 6 occurs 2. A sum more than 8 occurs

Solution:

Sample space by rolling two dice is
{ (1 , 1), (1 , 2), (1 , 3), (1 , 4), (1 , 5), (1 , 6),
(2 , 1), (2 , 2), (2 , 3), (2 , 4), (2 , 5), (2 , 6),
(3 , 1), (3 , 2), (3 , 3), (3 , 4), (3 , 5), (3 , 6),
(4 , 1), (4 , 2), (4 , 3), (4 , 4), (4 , 5), (4 , 6),
(5 , 1), (5 , 2), (5 , 3), (5 , 4), (5 , 5), (5 , 6),
(6 , 1), (6 , 2), (6 , 3), (6 , 4), (6 , 5), (6 , 6) }

1. Let A = event containing pairs having a sum 6
A = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)}
No of sample points in the sample space = 36

So, $$P(A) = \frac{\text{Number of sample points in A}}{\text{Total number of sample points}}$$

$$P(A) = \frac{5}{36}$$

2. Let B = event containing pairs having a sum more than 8.
B = {(3, 6), (4, 5), (4, 6), (5, 4), (5, 5), (5, 6), (6, 3), (6, 4), (6,5), 6, 6)}

$$P(B) = \frac{\text{no. of sample points in B}}{\text{total no. of sample points in the sample space}}$$

$$P(B) = \frac{10}{36} = \frac{5}{18}$$

Example 4: A box contains 6 red and 4 yellow balls. Two balls are drawn at random from the box. Find the probability that.

1. Both are yellow 2. One is red and one yellow

Solution: Total no. of ways in which two balls are drawn from 10 is $\binom{10}{2} = 45$.

Because, $$\binom{10}{2} = \frac{10!}{2!(10-2)!}$$

$$= \frac{10 \times 9 \times 8!}{2 \times 8!} = 45$$

1. No of ways that both drawn balls are yellow as $\binom{4}{2} = 6$

Probability that both are yellow balls $= \frac{6}{45} = \frac{2}{15}$

2. No. of ways that on red ball is drawn $= \binom{6}{1} = 6$

No. of ways that one yellow ball is drawn $\binom{4}{1} = 4$

No. of ways that one red and one yellow ball is drawn $= 6 \times 4 = 24$

Probability that one is red and other is yellow $= \frac{24}{45} = \frac{8}{15}$

Exercise 6.1

Q.1: A fair coin is tossed twice. What is the probability that we get.
(i) At least one head (ii) One head

Q.2: A perfect die is rolled, find the probability that it shown an even number.

Q.3: What is the probability of getting at least one tail, when three coins are tossed.

Q.4: If a dice is rolled. What is the probability that
(i) An even no appears
(ii) An event no divisible by 3 appears

Q.5: If two dice are rolled. What is the probability that.
(i) A sum greater than 9 appears
(ii) Six appears on none of them
(iii) A sum divisible by 5 appears
(iv) Six appears on both of them
(v) A sum of 7 appears.

Q.6: Three dices are rolled. What is the probability that
(i) A sum of 11 appears
(ii) A sum of 15 appears

Q.7: A card is drawn at random from a well shuffled pack of 52 cards. What is the probability that it is.
(i) jack (ii) red card
(iii) Pictured card (iv) diamond

Q.8: A letter is chosen at random from the word "Magneto hydrodynamics". What is the probability that it is;
(i) The letter h (ii) A vowel (iii) Consonant

Q.9: There are 9 apples in a bag, 5 red and 4 white. At random two apples are drawn. What is the probability that:
(i) Both are red (ii) Both are white
(ii) One is red and other is white

Q.10: If all the digits in all the dates of May in a calendar are shuffled well. What is the probability that randomly drawn digit is.
(i) 2 (ii) 3

Answers

Q.1: $\frac{3}{4}$, $\frac{1}{2}$ **Q.2:** $\frac{1}{2}$ **Q.3:** $\frac{7}{8}$ **Q.4:** $\frac{1}{2}, \frac{1}{6}$

Q.5: $\frac{1}{6}, \frac{25}{36}, \frac{7}{36}, \frac{1}{36}, \frac{1}{6}$ **Q.6:** $\frac{27}{216}$, $\frac{5}{108}$ **Q.7:** $\frac{1}{13}, \frac{1}{2}, \frac{3}{13}, \frac{1}{4}$

Q.8: $\frac{1}{20}, \frac{3}{10}, \frac{7}{10}$ **Q.9:** $\frac{5}{18}, \frac{1}{6}, \frac{5}{9}$ **Q.10:** $\frac{13}{53}$, $\frac{5}{53}$

6.6 Laws of Probability:

Following are some of the basic rules for calculation of probability:

6.6.1 Addition Law for Mutually Exclusive Events:

Theorem 1:

If A and B are two mutually exclusive events. Probability for the happening of A or B is the sum of their individual probability.

i.e. $P(A \cup B) = P(A) + P(B)$

Proof:

Let the sample points in the sample space = n

Sample points in the event A = ℓ

Sample points in the event B = m

Then. $P(A) = \frac{\ell}{n}$ and $P(B) = \frac{m}{n}$

Now sample points in A or B = $\ell + m$

And $$P(A \cup B) = \frac{\ell + m}{n}$$

$$P(A \cup B) = \frac{\ell}{n} + \frac{m}{n}$$

$$= P(A) + P(B)$$

This is the proof for two events, we can also prove this theorem for more than two mutually exclusive events.

Example 5:

Two fair dice are rolled. What is probability that sum is 7 or 9.

Solution:

We know (from the def. of sample space) that there are 36 sample points in the sample space.

Let A = event containing pairs (sample points) whose sum is 7.

$= \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

$$P(A) = \frac{6}{36} = \frac{1}{6}$$

And B = event containing pairs with sum 9

$= \{(3, 6), (4, 5), (5, 4), (6, 3)\}$

$$P(B) = \frac{4}{36} = \frac{1}{9}$$

Now as we know that events are mutually exclusive

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \frac{1}{6} + \frac{1}{9} \\ &= \frac{9+6}{54} = \frac{15}{54} = \frac{5}{18} \end{aligned}$$

6.6.2 Addition Law for not Mutually Exclusive Events :

Theorem 2:

If A and B are not mutually exclusive events then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof:

Given that A and B are not mutually exclusive events.

From the figure we can see that A and B – (A ∩ B) are two mutually exclusive events. Now B and A∪B can be written as

$$B = (A \cap B) \cup \{B - (A \cap B)\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

And $$(A \cup B) = A \cup \{B - (A \cap B)\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Equation (i) becomes:

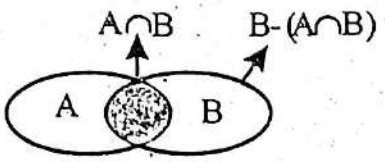
$$P(B) = P[(A \cap B) \cup \{B - (A \cap B)\}]$$

$$P(B) = P(A \cap B) + P\{B - (A \cap B)\} \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

$$\Rightarrow \quad P\{B - (A \cap B) = P(B) - P(A \cap B) \ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

Equation (ii) gives

$$\begin{aligned} P(A \cup B) &= P[A \cup \{B - (A \cap B)\}]) \\ &= P(A) + P\{B - (A \cap B)\} \end{aligned}$$



Putting value from (iv) we get

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This is our required proof for not mutually exclusive events

Example 6:

A student goes to a book depot. Probability that he buys a ball point is 0.3, he buys a pen is 0.75 and he buys both is 0.6.

What is the probability that he buys a ball point or a pen or both?

Solution:

Let A = event that he buys a ball point

B= event that he buys a pen

These are not mutually exclusive events because he can buy both of the items. So,

We have $P(A) = 0.3$, $P(B) = 0.75$ and $P(A \cap B) = 0.6$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.75 - 0.6$$
$$= 0.45$$

Independent events :

In probability theory to say that two events are independent means that the occurrence of one does not affect the probability of the other.

i.e. An event that is not affected by previous event.

For example, Tossing a coin, Heads or Tails is not affected by previous tosses.

6.7. Product Theorem of Probability or Multiplication Law for Independent Events:

Theorem: The probability of joint occurrence of two or more independent events is equal to the product of their respective probabilities.

Let we have two independent events A and B then their joint probability is equal to the product of their probabilities.

i.e. $P(A \cap B) = P(A) \cdot P(B)$; $P(A) \neq 0$, $P(B) \neq 0$

For example, when a single die is rolled twice in succession.

Probability of an outcome twice in succession is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

6.8 Conditional Probability :

Definition: If A and B are two events then conditional probability is defined as probability of happening of the event B given that A has already occurred and is denoted by $P\left(\frac{B}{A}\right)$

i.e., $$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)}$$

e.g. A fair die is rolled , find the probability that the face is even given that the face is less than 4.

Since , $S = \{1, 2, 3, 4, 5, 6\}$

Here , $A = \{2, 4, 6\}$, $B = \{1, 2, 3\}$

$A \cap B = \{2\}$

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = = \frac{1/6}{3/6} = \frac{1}{3}$$

Which is the required conditional probability .

Dependent events :

Two events A and B are called dependent if the occurrence of A depends upon the occurrence of B.

Now we study a theorem related to the conditional probability.

6.9 Product Theorem for dependent Events :

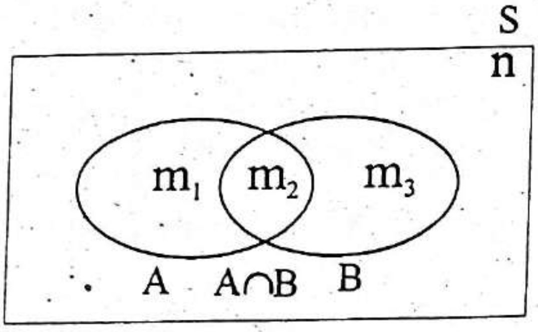
Theorem: If A and B are two dependent events then probability that both will occur is $P(A \cap B) = P(A) \times P(B/A)$

Proof: Given that A and B are not independent events, we show them by the following figure.

Let S be a sample space having n sample points. A has m_1, $A \cap B$ has m_2 and B has m_3 sample points.

Then $P(A \cap B) = \frac{m_2}{n}$

We can write, $P(A \cap B) = \frac{m_2}{m_1} \frac{m_1}{n}$ ....(i)



But, $P(A) = \frac{m_1}{n}$

And $\frac{m_2}{m_1}$ = Conditional Probability of B given that A has already occurred = $P\left(\frac{B}{A}\right)$

So (i) can be written as

$$P(A \cap B) = P\left(\frac{B}{A}\right) . P(A)$$

Gives the proof

Example 7: A fair die is rolled once. What is the probability of getting a number > 3 given that we have already got an even number.

Solution:

Sample space $S = \{1, 2, 3, 4, 5, 6\}$

Let A = Set of numbers greater than 3 = $\{4, 5, 6\}$

And B = Set of even numbers $\{2, 4, 6\}$

$$A \cap B = \{4, 6\}$$

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$

Where $P(B) = \frac{3}{6} = \frac{1}{2}$

$$P(A \cap B) = \frac{2}{6} = \frac{1}{3}$$

$$P\left(\frac{A}{B}\right) = \frac{1/3}{1/2} = \frac{2}{3}$$

Exercise 6.2

Q.1: Two fair dice are rolled. Find the Probability that the sum is 6 or 8.

Q.2: Find the probability that an integer selected at random from 10 to 100 inclusive is a multiple of 5 or 9.

Q.8: Two coins are tossed. What is the probability that two heads appear, given that there is at least one head?

Q.3: An integer is chosen at random from first 200 positive integers. What is the probability that the integer chosen is divisible by 6 or 8.

Q.4: A pair of dice is rolled. Let A denote the event "the sum shown is 6" and the event "B" two dice show the same number.

Find (1) $P\left(\frac{A}{B}\right)$ (2) $P\left(\frac{B}{A}\right)$

Q.7: From a well shuffled pack of 52 two cards are drawn in succession and without replacement. Find the probability that first card is a king and second a queen.

Q.4: A fair dice is rolled once. You win if the outcome is either even or divisible by 3. What is the probability that you win the game.

Q.6: Find P(B) in following cases:
A and B are mutually exclusive given that

$P(A) = 0.5$ and $P(A \cup B) = 0.6$.

Q.5: A die is thrown, find the probability that the dots on the top are prime numbers or odd numbers.

Q.10: Two fair dice are rolled , what is the probability that the sum is 7 or 9 ?

Answers

Q.1: $\frac{5}{18}$ **Q.2:** $\frac{27}{91}$ **Q.3:** 0.25 **Q.4:** $\frac{2}{3}$ **Q.5:** $\frac{2}{3}$

Q.6: 0.1 **Q.7:** $\frac{16}{2652}$ **Q.8:** $\frac{1}{3}$ **Q.9:** $\frac{1}{6}$, $\frac{1}{5}$

Short Questions

Write the short answers of the following :

Q.1: If a die is rolled once, what is the probability of getting an even number.

Q.2: A card is drawn at random from a deck of cards. Find the probability, of getting a diamond.

Q.3: If two coins are tossed, find the probability that only one head

Q.4: Tickets numbered 1 to 20 are mixed up and then a ticket is drawn at random. What is the probability that ticket drawn has a number multiple of 3 or 5.

Q.5: If a dice is rolled once, what is the probability of getting a 4?

Q.6: If two cones are tossed, what is the probability of getting two heads?

Q.7: If two dice are rolled, what is the probability of getting the same number?

Q.8: If two dice are rolled, find the probability that the sum is 7.

Q.9: Write down formula to find the probability of two not mutually exclusive events.

Q.10: What are mutually exclusive events?

Q.11: Define a sample space.

Answers

Q1. $\frac{1}{2}$ Q2. $\frac{1}{4}$ Q3. $\frac{3}{4}$ Q4. $\frac{9}{20}$ Q5. 1/6

Q6. 1/4 Q7. 1/6 Q8. 1/6

Q9. P(AUB)= P(A) +(P(B) – (P(A∩B)

Q10. When there is nothing common between different events, then they are called mutually exclusive events.

Q11. It is the set of all the outcomes of an experimental.

For example when a coin is tossed, sample space is $\{H,T\}$.

Find the mod

Objective Types Questions

Q.1: Encircle the correct one of the given answers in each item.

S.P 1--- A process of obtaining an observation is called
[✓ (a) an experiment] (b) Trial (c) outcome (d) an event

2--- A single performance of an experiment is called
(a) an event (b) Trial (c) outcome (d) sample space

3--- The result obtained from an experiment or a trail is called
(a) sample space (b) an event (c) outcome (d) population

4--- The collection of all possible outcomes of an experiment is called
(a) population (b) an event (c) trial (d) sample space

5--- Each element of a sample space is called
(a) trial (b) sample point (c) experiment (d) an event

6--- When each outcome of the sample space is as likely to occur as any other, then these are called
(a) mutually exclusive events (b) exhaustive events
(c) compound events (d) equally likely events

7--- Toss of a fair coin is an example of
(a) equally likely events (b) independent events
(c) simple events (d) dependent events

8--- Toss of a fair coin is an example of
(a) independent events (b) dependent events
(c) mutually exclusive events (d) not mutually exclusive events

9--- If the entire sample space is used then it is called
(a) not mutually exclusive (b) equally likely
(c) mutually exclusive (d) exclusive events

S.P 10--- A perfect coin is tossed, what is the probability that it shows head
[✓ (a) 1/2] (b) zero (c) 1 (d) ¾

11--- A perfect coin is tossed, what is the probability that it shows tail
(a) zero (b) 1/2 (c) 1 (d) 3/4

12--- A perfect die is rolled, its sample space consists of total elements
(a) 4 (b) 3 (c) 6 (d) 1

S.P 13--- The probability of occurring once when a perfect die is rolled
(a) 5/6 (b) 2/3 (c) 1/3 [✓ (d) 1/6]

14--- From a well shuffled pack of 52 cards, a card is drawn at random, the probability that it is king card
(a) 1/13 (b) 1/26 (c) 1/14 (d) 1/52

15--- From a well shuffled pack of 52 cards, a card is drawn at random, the probability that it is spade card
(a) 1/13 (b) 1/4 (c) 1/26 (d) 1/52

16--- From a well shuffled pack of 52 cards a card is drawn at random, the probability that it is red card
(a) 1/52 (b) 1/13 (c) 1/2 (d) 1/4

17--- From a well shuffled pack of 52 cards a card is drawn at random, the probability that it is pictured card
(a) 1/52 (b) 1/2 (c) 1/4 (d) 3/13

18--- Law of addition for non-mutually exclusive events
(a) $P(A\cup B) = P(A) + P(B) - P(A\cap B)$
(b) $P(A\cup B) = P(A) + P(B) + P(A\cap B)$
(c) $P(A\cup B) = P(A) - P(B) + P(A\cap B)$
(d) $P(A\cup B) = P(A) - P(B) - P(A\cap B)$

19--- Law of addition for mutually exclusive events
(a) $P(A\cup B) = P(A) - P(B)$ (b) $P(A\cup B) = P(A) + P(B)$
(c) $P(A\cup B) = P(A) . P(B)$ (d) $P(A\cup B) = \frac{P(A)}{P(B)}$

20--- Multiplication Law for independent events
(a) $P(A\cap B) = P(A) . P(B)$ (b) $P(A\cap B) = P(A) P(B/A)$
(c) $P(A\cap B) = P(B) . P(A/B)$ (d) $P(A\cap B) = P(B) . P(B/A)$

Answers

Q1.

1.	a	**2.**	b	**3.**	c	**4.**	d	**5.**	b
6.	d	**7.**	a	**8.**	c	**9.**	d	**10.**	a
11.	b	**12.**	c	**13.**	d	**14.**	a	**15.**	b
16.	c	**17.**	d	**18.**	a	**19.**	b	**20.**	a

Chapter 7

Integrations

7.1 Introduction:

In the earlier chapters of this text, the problems to find the derivative of ordinary functions have been considered. The inverse process of finding the function when its derivative is given will be considered in this chapter. We shall find that this inverse process, called integration, enables us to find areas, volumes, solution of differential equations, and many other quantities useful in solving physical problems. Accordingly the process of differentiation and the inverse process called integration have played a very important role in the development of the physical sciences and engineering.

7.2 Indefinite Integration:

The process of finding a function whose derivative is known is called integration and the required function is called an integral or an anti-derivative of the given function. *i.e.*

$$\text{If} \quad \frac{d}{dx}F(x) = f(x)$$

$$\text{Then} \quad \int f(x)\,dx = F(x) + c$$

The left hand member is read as "integral of $f(x.)$ with respect to x". the symbol $\int$ is an integral sign, $f(x)$ is the integrand , c is the constant of integration, and $F(x) + c$ is the indefinite integral.

Note: That if $F(x)$ is an integral of $f(x)$ with respect to x, then $F(x) + c$ is also such an integral of $f(x)$, where c is any constant, since the derivative of any constant is zero that is.

$$\frac{d}{dx}[F(x) + c] = \frac{d}{dx}F(x) + \frac{d}{dx}(c)$$

$$= \frac{d}{dx}F(x) + 0 \quad = f(x)$$

Thus a function whose derivative is given is not completely determined, since it contains an arbitrary additive constant, the constant of integration. It is for this reason that the function $\int f(x)\,dx$ is referred to as the indefinite integral of $f(x)$..

It can be shown that two functions which have the same derivative differ at the most by a constant, thus if $F(x)$ is an integral of $f(x)$, all integrals of $f(x)$ are included in the set of $F(x) + c$, where c is any constant. In many applications of integration, certain information given in the problem (often referred to as an initial or boundary condition) uniquely specify the constant of integration.

7.3 Basic Rules of Integration:

Rule -I:

The differential operator $\frac{d}{dx}$ and $\int dx$ cancel each other, they are inverse of each other, that is

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

Rule – II:

A constant multiple of an integrand is taken out from integral sign without affecting the value of integral, that is

$$\int C f(x)\,dx = C\int f(x)\,dx$$

Rule – III:

The integral of the sum (or difference) of a finite number of functions is equal to the sum (or difference) of their integrals, that is,

$$\int[f(x) \pm g(x) + u(x) \pm \ldots]\,dx = \int f(x)\,dx \pm \int g(x)\,dx \pm \int u(x)\,dx \pm \ldots\ldots\ldots$$

7.4 Fundamental Formulas of integration:

We make start with the following integral which need no proofs. We can verify them by differentiating the R.H.S. in each case.

Algebraic Functions:

1. $\int dx = x + c$

2. $\int c f(x)\,dx = c\int f(x)\,dx$

3. $\int [f(x) \pm g(x) \pm \ldots]\,dx = \int f(x)\,dx \pm \int g(x)\,dx \pm \ldots\ldots\ldots$

4. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \qquad n \neq -1$

5. $\int u^n\,du = \frac{u^{n+1}}{n+1} + c, \; n \neq -1$, where $u = f(x)$ is a differentiable function of x.

6. $\int (ax+b)^n\,dx = \frac{1}{a}\,\frac{(ax+b)^{n+1}}{n+1} + c, \qquad n \neq -1$

Logarithmic and Exponential Functions:

7. $\int \frac{dx}{x} = \ln x + c$

8. $\int \frac{dx}{ax+b} = \frac{1}{a} \ln(ax+b) + c$

9. $\int e^{x}\, dx = e^{x} + c$

10. $\int e^{ax}\, dx = = \frac{e^{ax}}{a} + c$

Integrals giving inverse trigonometric functions

11. $\int \frac{dx}{1+x^2} = \tan^{-1} x + c$ or $-\cot^{-1} x + c$

12. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$ or $-\cos^{-1} x + c$

13. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$ or $-\operatorname{cosec}^{-1} x + c$

These can be easily verified by differentiating the right hand members.

Note: That **$\sin^{-1}x$ = arc sin x.** similar relation is used for others ,

also $\sin^{-1}x \neq (\sin x)^{-1}$

Example 1:

$\int (3x^2 + 2x + 1)\, dx$

Solution:

$$\int (3x^2 + 2x + 1)\, dx = 3\int x^2\, dx + 2\int x\, dx + \int dx$$

$$= 3\frac{x^{2+1}}{3} + 2\frac{x^{1+1}}{2} + x + c$$

$$= x^3 + x^2 + x + c$$

Example 2:

(i) $\int \frac{dx}{\sqrt{x}}$ (ii) $\int \frac{1}{x^2}\, dx$

Solution **(i)**

$$\int \frac{dx}{\sqrt{x}} = \int x^{-1/2}\, dx$$

$$-\frac{1}{2}+1$$

$$= \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$\frac{-1+2}{2} \Rightarrow \frac{1}{2}$$

$$= 2\sqrt{x} + c$$

(ii) $\int \frac{1}{x^2}\, dx = \int x^{-2}\, dx = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x}$

Example 3:

$$\int \sqrt{x}\ (x^2 + 3x + 2)\, dx$$

Solution:

$$\int \sqrt{x}\ (x^2 + 3x + 2)\, dx = \int x^{1/2}\, (x^2 + 3x + 2)\, dx$$

$$= \int (x^{5/2} + 3x^{3/2} + 2x^{1/2})\, dx$$

$$= \int x^{5/2}\, dx + 3 \int x^{3/2}\, dx + 2 \int x^{1/2}\, dx$$

$$= \frac{x^{7/2}}{7/2} + 3\,\frac{x^{5/2}}{5/2} + 2\,\frac{x^{3/2}}{3/2} + C$$

$$= \frac{2}{7} x^{7/2} + \frac{6}{5} x^{5/2} + \frac{4}{3} x^{3/2} + C$$

Example 4:

$$\frac{1}{x} + \frac{\cancel{x}}{\cancel{x}}$$

$$\int \frac{1 + x}{x}\, dx$$

$$\frac{1}{x} + 1$$

Solution:

$$\int \frac{1 + x}{x}\, dx = \int \frac{1}{x}\, dx + \int dx \Rightarrow \ln x + x + c$$

Example 5:

Find $\int (e^x + e^{2x} + e^{3x})\, dx$

Solution:

$$\int (e^x + e^{2x} + e^{3x})\, dx = \int e^x\, dx + \int e^{2x}\, dx + \int e^{3x}\, dx$$

$$= e^x + \frac{1}{2} e^{2x} + \frac{1}{3} e^{3x} + c$$

Exercise 7.1

Q.1: $\int \sqrt{x}\, dx$

Q.2: $\int x^{-7}\, dx$

Q.3: $\int x^{-3/4}\, dx$

Q.4: $\int \sqrt{x} \left(x^5 + \frac{1}{x}\right) dx$

Q.5: $\int \frac{ax + bx^{-3} + cx^{-7}}{x^{-2}}\, dx$

Q.6: $\int \frac{x^3 + 1}{x^5}\, dx$

Q. 7: $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$

Q.8: $\int \left(3t - \sqrt{3t} + \frac{1}{\sqrt{t}}\right) dt$

Q.9: $\int \left(12 - \frac{2}{t^2} + \frac{4}{t^3}\right) dt$

Q.10: $\int \left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right) dx$

Q.11: $\int (ax^n + bx^m)\, dx$

Q.12: $\int \frac{x^4}{x+1}\, dx$

Q.13: $\int \frac{x^3 - 8}{x + 2}\, dx$

Q.14: $\int \frac{dx}{2x + 3}$

Q.15: $\int \frac{1}{\sqrt{x + a} + \sqrt{x + b}}\, dx$

Q.16: $\int \frac{1}{\sqrt{1 + x} - \sqrt{x}}\, dx$

Q.17: $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

Q.18: $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

Q.19: $\int \frac{1}{2}\left(e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}\right) dx$

Q.20: $\int (e^x + e^{-x})^2\, dx$

Chapter # 7

Answers 7.1

Q.1: $\frac{2}{3}x^{3/2}+c$

Q.2: $-\frac{1}{6x^6}+c$

Q.3: $4x^{1/4}+c$

Q.4: $\frac{2}{13}x^{13/2}+2x^{1/2}+c$

Q.5: $\frac{1}{4b}[ax^4+4b\ \mathrm{In}\ x-cx^{-4}]$

Q.6: $-\frac{1}{x}-\frac{1}{4x^4}+c$

Q.7: $\frac{2}{3}x^{3/2}+2x^{1/2}+c$

Q.8: $\frac{3}{2}t^2-\frac{2}{\sqrt{3}}t^{3/2}+2\sqrt{t}+c$

Q.9: $12t+\frac{2}{t}-\frac{2}{t^2}+c$

Q.10: $\frac{1}{4}x^4+\frac{1}{2}x^2+\ln x-\frac{1}{2x^2}+c$

Q.11: $\frac{ax^{n+1}}{n+1}+\frac{bx^{m+1}}{m+1}+c$

Q.12: $\frac{1}{4}x^4-\frac{1}{3}x^3+\frac{1}{2}x^2-x+\mathrm{In}\ (x+1)+c$

Q.13: $\frac{1}{3}x^3-x^2+4x-16\ \mathrm{In}\ (x+2)+c$

Q.14: $\frac{1}{2}\mathrm{In}\ (2x+3)+c$

Q.15: $\frac{2}{3(a-b)}[(x+a)^{3/2}-(x+b)^{3/2}]+c$

Q.16: $\frac{2}{3}[(1+x)^{3/2}+x^{3/2}]+c$

Q.17: $\frac{1}{2}x^2+2x+\ln x+c$

Q.18: $2e^{\sqrt{x}}+c$

Q.19: $e^{\frac{1}{2}x}+e^{-\frac{1}{2}x}+C$

Q.20: $\frac{1}{2}e^{2x}-\frac{1}{2}e^{-2x}+2x+c$

7.5 Integration of Trigonometric Functions:

1. $\int \sin x\ dx = -\cos x+c$
2. $\int \cos x\ dx = \sin x+c$
3. $\int \sec^2 x\ dx = \tan x+c$
4. $\int \mathrm{cosec}^2 x\ dx = -\cot x+c$
5. $\int \sec x\tan x\ dx = \sec x+c$
6. $\int \mathrm{cosec}\ x\cot x\ dx = -\mathrm{cosec}\ x+c$
7. $\int \tan x\ dx = -\mathrm{In}\cos x+c = \mathrm{In}\sec x+c$
8. $\int \cot x\ dx = \mathrm{In}\sin x+c$

2 sinx cosx
sin x

9. $\int \sec x \; dx = \ln(\sec x + \tan x) + c$

10. $\int \operatorname{cosec} x \; dx = \ln(\operatorname{cosec} x - \cot x) + c$

The following illustrations suggest procedures. Using (1), we get

$$\int \sin ax \, dx = \frac{1}{a}\int \sin ax\,(a\,dx) = \frac{1}{a}\cos ax + c$$

Note: That u = ax, du = a dx; therefore we have to supply under the integral sign to complete du before the formula would apply

Example 1:

(i) $\int (\sin x - \cos x)\,dx$ (ii) $\int 2\sin x\cos x\,dx$

(iii) $\int \frac{\sin 2x}{\sin x}\,dx$ (iv) $\int \frac{\sin 2x}{\cos x}\,dx$

Solution:

(i) $\int (\sin x\,dx - \cos x)\,dx$

$= \int \sin x\,dx - \int \cos x\,dx$

$= -\cos x - \sin x + c$

(ii) $\int 2\sin x\cos x\,dx = \int \sin 2x\,dx = -(1/2)\cos 2x + c$

(iii) $\int \frac{\sin 2x}{\sin x}\,dx = \int \frac{2\sin x\cos x}{\sin x}\,dx = 2\int \cos x\,dx$

$= 2\sin x + c$

(iv) $\int \frac{\sin 2x}{\cos x}\,dx = \int \frac{2\sin x\cos x}{\cos x}\,dx = 2\int \sin x\,dx$

$= -2\cos x + c$

Example 2:

$$\int \frac{1}{1+\cos x.}\,dx$$

Solution:

$$\int \frac{1}{1+\cos x.}\,dx$$

Multiplying and dividing by (1 – cos x), we get

$$\int \frac{1}{1+\cos x}\cdot\frac{1-\cos x}{1-\cos x}\,dx$$

$$= \int \frac{1-\cos x}{1-\cos^2 x}\,dx$$

$$= \int \frac{1-\cos x}{\sin^2 x}\,dx$$

$$= \int \frac{1}{\sin^2 x}\,dx - \int \frac{\cos x}{\sin^2 x}\,dx$$

$$= \int \operatorname{cosec}^2 x\,dx - \int \operatorname{cosec} x \cot x\,dx$$

$$= -\cot x + \operatorname{cosec} x + c$$

Example 3:

(i) $\int \cot^2 x\,dx$ (ii) $\int \cot(3x+5)\,dx$ (iii) $\int \tan^2 x \operatorname{cosec}^2 x\,dx$

Solution:

(i) $\int \cot^2 x\,dx = \int(\operatorname{cosec}^2 x - 1)\,dx$

$$= \int \operatorname{cosec}^2 x\,dx - \int dx$$

$$= -\cot x - x + c$$

(ii) $\int \cot(3x+5)\,dx = \frac{1}{3}\int \cot(3x+5)(3dx)$

$$= \frac{1}{3}\ln \sin(3x+5) + c$$

(iii) $\int \tan^2 x \operatorname{cosec}^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\cdot\frac{1}{\sin^2 x}\,dx$

$$= \int \sec^2 x\,dx = \tan x + c$$

Example.4

(i) $\int \sec 3x \tan 3x\,dx$ (ii) $\int \frac{2}{\cos^2 2x}\,dx$

Solution :

(i) $\int \sec 3x \tan 3x \, dx = \frac{1}{3} \int \sec 3x \tan 3x \, (3\,dx)$

$$= \frac{1}{3} \sec 3x + c$$

(ii) $\int \frac{2}{\cos^2 2x} dx = \int \sec^2 2x \, (2\,dx) = \tan 2x + c$

7.6 Certain Trigonometric Integrals

(a) $\int \text{Sin}^m u \, \text{Cos}^n u du.$

When both m and n are even and positive.

In this case, the half-angle formulas are used to lower the degree of the expression. These formulas (which the student should have memorized well) are

$$\sin^2 u = \frac{1 - \cos 2u}{2} \qquad \cos^2 u = \frac{1 + \cos 2u}{2}$$

The method of reduction is shown in the following examples:

Example 5:

Find $\int \sin^2 x \, dx$

Solution: $\int \sin^2 x \, dx$

$$= \int \left(\frac{1 - \cos 2x}{2} \right) dx$$

$$= \frac{1}{2} [\int dx - \int \cos 2x \, dx]$$

$$= \frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right] + c$$

Example 6:

Find $\int \sin^2 x \cos^2 x \, dx$

Solution:

$$\int \sin^2 x \cos^2 x \, dx = \int (\sin x \, . \cos x)^2 \, dx$$

$$= \int \frac{1}{4} (2 \sin x \,.\, \cos x)^2 \, dx = \frac{1}{4} \int (\sin 2x)^2 \, dx$$

$$= \frac{1}{4} \int (1 - \cos^2 2x) \, dx$$

To this last integral we again apply the half angle formula, and get

$$\int \sin^2 x \cos^2 x \, dx = \frac{1}{4} \int (1 - \frac{1 + \cos 4x}{2}) \, dx$$

$$= \frac{1}{4} \int \frac{1 - \cos 4x}{2} \, dx$$

2 - 1 - cos4

1 - cos4x

$$= \frac{1}{8} \int (1 - \cos 4x) dx$$

$$= \frac{1}{8} \left(x - \frac{1}{4} \sin 4x \right) + C$$

Example 7:

Find $\int \sin^4 x \, dx$

Solution:

$$\int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx = \frac{1}{4} \int (1 - \cos 2x)^2 \, dx$$

$$= \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{4} \int (1 - 2 \cos 2x + \frac{1 + \cos 4x}{2}) \, dx$$

$$= \frac{1}{4} \int \left(\frac{3 - 4 \cos 2x + \cos 4x}{2} \right) dx$$

$$= \frac{1}{8} \int (3 - 4 \cos 2x + \cos 4x) \, dx$$

$$= \frac{1}{8} \left(3x - 4 \,.\, \frac{\sin 2x}{2} + \frac{\sin 4x}{4} \right)$$

$$= \frac{1}{8} \left(3x - 2 \sin 2x + \frac{\sin 4x}{4} \right) + c$$

7.7 Extension Rules

Integration of the function f(bx + c)

(i) $\int (bx+c)^n\,dx = \dfrac{(bx+c)^{n+1}}{b(n+1)+c}$

(ii) $\int \dfrac{1}{(bx+c)}\,dx = \dfrac{1}{b}\,\ell n\,(bx+c)+c$

(iii) $\int e^{bx+c}\,dx = \dfrac{1}{b}\,e^{bx+c}+c$

(iv) $\int \text{Sin}\,(bx+c)\,dx = -\dfrac{1}{b}\,\text{Cos}\,(bx+c)+c$

(v) $\int \text{Cos}\,(bx+c)\,dx = \dfrac{1}{b}\,\text{Sin}\,(bx+c)+c$

(vi) $\int \text{Sec}^2\,(bx+c)dx = \dfrac{1}{b}\,\tan\,(bx+c)+c$

(vii) $\int \text{Sec}(bx+c)\,\text{Tan}\,(bx+c)\,dx = \dfrac{1}{b}\,\text{Sec}\,(bx+c)+c$

(viii) $\int \text{Cosec}^2\,(bx+c)\,dx = -\dfrac{1}{b}\,\text{Cot}(bx+c)+c$

(ix) $\int \text{Cosec}(bx+c)\,\text{Cot}(bx+c) = -\dfrac{1}{b}\,\text{Cosec}\,(bx+c)+c$

Exercise 7.2

Q.1: $\int \cos^2 x\,dx$

Q.2: $\int \tan^2 x\,dx$

Q.3: $\int \cos^4 2x\,dx$

Q.4: $\int \dfrac{\cos^2 x - \sin^2 x}{\sin 2x}\,dx$

Q.5: $\int \dfrac{dx}{1+\sin x}$

Q.6: $\int \dfrac{dx}{1-\cos x}$

Q.7: $\int \sec^2 x\,\text{cosec}^2 x\,dx$

Q.8: $\int \dfrac{a\sin^3 x + b\cos^3 x}{\sin^2 x\cos^2 x}$

Q.9: $\int \dfrac{\sin^2 x - \cos^2 x}{\sin^2 x\cos^2 x}$

Q.10: $\int (\tan x + \cot x)^2\,dx$

Q.11: $\int (\sin x - \cos x)^2\,dx$

Q.12: $\int \text{Sec}^4 x\,dx$

Answers 7.2

Q.1: $\frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)+C$

Q.2: $\tan x - x + c$

Q.3: $\frac{1}{8}(3x+\sin 4x+\frac{1}{8}\sin 8x)+c$

Q.4: $\frac{1}{2}\ell n \sin 2x + c$

Q.5: $\tan x - \sec x + C$

Q.6: $-\cot x - \operatorname{cosec} x + C$

Q.7: $\tan x - \cot x + C$

Q.8: $a \sec x - b \operatorname{cosec} x + C$

Q.9: $\tan x + \cot x + C$

Q.10: $\tan x - \cot x + C$

Q. 11: $x+\frac{1}{2}\cos 2x + C$

Q. 12: $\tan x + \frac{1}{3}\tan^3 x + c$

7.8 Useful Generalization of Some Standard Formulas:

Generalization of $\int x^n dx$ and $\int \frac{1}{x} dx$

Rule I:

$$\int [f(x)]^n \, f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

Rule II:

$$\int \frac{f'(x)}{f(x)}\, dx = \text{In}\,[f(x)] + c$$

Example 1 :

$$\int \frac{(x+1)}{(x^2+2x+7)^{-\frac{3}{2}}}\, dx$$

Solution:

$$\int \frac{(x+1)}{(x^2+2x+7)^{-\frac{3}{2}}}\, dx = \frac{1}{2}\int (x^2+2x+7)^{3/2}(2x+2)dx$$

Here $f(x) = x^2+2x+7$

$f'(x) = 2x+2$

$\therefore$ Using Rule-I

$$= \frac{1}{2}\frac{(x^2+2x+7)^{\frac{3}{2}+1}}{\frac{3}{2}+1}+c$$

$$= \frac{1}{2}\frac{(x^2+2x+7)^{\frac{5}{2}}}{\frac{5}{2}}+c$$

$$= \frac{1}{5}(x+2x+7)^{\frac{5}{2}}+c$$

Example 2:

$$\int \text{Sin}^3 x \cos x\, dx$$

Solution:

$$\int \text{Sin}^3 x \cos x\, dx$$

Here $f(x) = \sin x$

$f'(x) = \cos x$

$\therefore$ Using Rule –I

$$= \frac{\sin^{3+1} x}{3+1}+c$$

$$= \frac{\sin^4 x}{4}+c$$

Example 3:

$$\int \tan x \sec^2 x\, dx$$

Solution:

$$\int \tan x \sec^2 x\, dx$$

Here $f(x) = \tan x$

$f'(x) = \sec^2 x$

$\therefore$ Using Rule –I

$$= \frac{\tan^2 x}{2}+c$$

Example 4:

$$\int (\sin x - \cos)^4 (\cos x + \sin x)\, dx$$

Solution:

$$\int(\sin x - \cos)^4 (\cos x + \sin x)\,dx$$

Here $f(x) = \sin x - \cos x$

$$f'(x) = \cos x + \sin x$$

$\therefore$ Using Rule –I

$$= \frac{(\sin x - \cos x)^{4+1} x}{4+1} + c$$

$$= \frac{(\sin x - \cos x)^5}{5} + c$$

Example 5:

(i) $\int \frac{dx}{x(Inx)^4}$ (ii) $\int \frac{\sqrt{1+\ln x}}{x}\,dx$

Solution:

(i) $\int \frac{dx}{x(Inx)^4}$

Here $f(x) = In\,x$

$$f'(x) = \frac{1}{x}$$

$$= \int In\,x)^{-4}\left(\frac{1}{x}\right)dx$$

$\therefore$ Using Rule –I

$$= \frac{(Inx)^{-4+1}}{-4+1} + c$$

$$= \frac{(Inx)^{-3}}{-3} + c$$

$$= -\frac{1}{3(Inx)^3} + c$$

Let $I = \int \frac{\sqrt{1+\ln x}}{x}\,dx = \int \frac{(1+\ln x)^{\frac{1}{2}}}{x}\,dx$

Here, $f(x) = 1 + \ln x$, $f'(x) = 1/x$

Using (Rule I) $\quad I = \dfrac{(1+\ln x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}$

So, $\quad I = \frac{2}{3}(1 + \text{ln}x)^{3/2} + c$

Example 6:

$$\int \frac{\cot x}{\text{In} \sin x}$$

Solution:

$$\int \frac{\cot x}{\text{In} \sin x}$$

Here $\quad f(x) = \text{In} \sin x$

$f'(x) = \cot x$

∴ Using Rule –II

$= \text{In} |\text{In} \sin x| + c$

Example 7:

$$\int \frac{1-\sin x}{x+\cos x} dx$$

Solution:

$$= \int \frac{1-\sin x}{x+\cos x} dx$$

Here $\quad f(x) = x + \cos x$

$f'(x) = 1 - \sin x$

∴ Using Rule II

$= \text{In} |(x + \sin x)| + c$

Example 8: Evaluate the following

(i) $\int \tan x \, dx$ (ii) $\int \cot x \, dx$

Solution:

(i) Let $I = \int \tan x \, dx$

$$= \int \frac{\sin x}{\cos x} dx$$

Here $f(x) = \cos x$

$f'(x) = - \sin x$

(ii) Let $I = \int \cot x \, dx$

$$= \int \frac{\cos x}{\sin x} dx$$

Here $f(x) = \sin x$

$f'(x) = \sin x$

$$= -\int \frac{-\sin x}{\cos x} dx$$

∴ Using Rule II i.e

$$\int \frac{f'(x)}{f(x)} dx = \ell n\, f(x) + c$$

$$= -\text{In}\,|\cos x| + c$$

$$= \text{In}|(\cos x)^{-1}| + c$$

$$= \text{In}\left|\frac{1}{\cos x}\right| + c$$

$$= \text{In}\,|\sec x| + c$$

Using Rule II i.e.

$$\int \frac{f'(x)}{f(x)} dx = \ell n\, f(x) + c$$

Hence

$$I = \int \frac{\cos x}{\sin x} dx$$

$$= \ell n \sin x + c$$

Example 9:

$$\int \frac{dx}{x(1 + \text{In}\, x)}$$

Solution:

$$= \int \frac{dx}{x(1 + \text{In}\, x)}$$

Here $f(x) = 1 + \text{In}\, x$

$f'(x) = \frac{1}{x}$

∴ Using Rule II

$$= \text{In}\,|(1 + \text{In}\, x)| + c$$

Example 10:

$$\int \frac{\sec x \operatorname{cosec} x}{\text{In} \tan x} dx$$

Solution:

$$= \int \frac{\sec x \operatorname{cosec} x}{\text{In} \tan x} dx$$

Here $f(x) = \text{In} \tan x$

$f'(x) = \sec x \operatorname{cosec} x$

∴ Using Rule II

$$= \text{In}\,|\text{In} \tan x| + c$$

7.9 Certain Trigonometric Integrals

We now consider trigonometric integrals of the form.

(a) $\int \sin^m u \cos^n u \, du$

(b) $\int \tan^m u \sec^n u \, du$

(c) $\int \cot^m u \operatorname{cosec}^n u \, du$

Under Type (a): $\int \sin^m u \cos^n u \, du$

Either m or n odd and positive,

If m is odd, we factor out sin u du and change the remaining even power of sine to powers of cosine by the identity.

$$\sin^2 u + \cos^2 u = 1$$

If n is odd, we factor out cos u du and change the remaining even power of cosine to powers of sine by the same identity.

Example 11:

Find $\int \sin^3 x \cos^5 x \, dx$

Solution:

Since m (=3) is odd and positive we have

$$\int \sin^3 x \cos^5 x \, dx = \int \sin^2 x \cos^5 x \sin x \, dx$$

$$= \int (1 - \cos^2 x) \cos^5 x \sin x \, dx$$

$$= -\int \cos^5 x (-\sin x) dx + \int \cos^7 x (-\sin x) dx$$

(by rule I)

$$= -\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + c$$

Example 12:

Find $\sin^4 2x \cos^5 2x \, dx$.

Solution:

Since n(=5) is odd and positive, we write

$$\int \sin^4 2x \cos^5 2x\, dx = \int \sin^4 2x \cos^4 2x \cos 2x\, dx$$

$$= \int \sin^4 2x(1 - \sin^2 2x)^2 \cos 2x\, dx$$

$$= \frac{1}{2} \int (\sin^4 2x - 2\sin^6 2x + \sin^8 2x)\,(2 \cos 2x\, dx)$$

$$= \frac{1}{10} \sin^5 2x - \frac{1}{7} \sin^7 2x + \frac{1}{18} \sin^9 2x + C$$

Under Type (b or c)

$\int \tan^m u \sec^n u\, du$ $\qquad$ $\int \cot^m u \operatorname{cosec}^n u\, du$

There are also Two cases:

Case – I: n is even and positive

We factor out $\sec^2 u\, du$ and change the remaining secants to tangents, using the identity.

$$\text{Sec}^2 u = 1 + \tan^2 u$$

Example 13:

Find $\int \tan x \sec^4 x\, dx$

Solution:

We write $\int \tan x \sec^4\, dx = \int \tan x\,(1 + \tan^2 x) \sec^2 x\, dx$

$$= \int (\tan x) \sec^2 x\, dx + \int (\tan x)^3 \sec^2 x\, dx$$

$$= \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + c$$

Case II: m is odd and positive:

We factor out $\sec u \tan u\, du$ and change the remaining even power of the tangents to secants, again using the identity $\tan^2 u = \sec^2 u - 1$.

Example 14: Find $\int \tan^3 x \sec^3 x\, dx$

Solution: We write

$$\int \tan^3 x \sec^3 x\, dx = \int \tan^2 x \sec^2 x\,(\tan x \sec x)dx$$

$$\int (\sec^2 x - 1) \sec^2 x\,(\tan x \sec x)dx$$

$$= \int (\sec^4 x - \sec^2 x)\,(\tan x \sec x)\,dx$$

$$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3}$$

Exercise 7.3

Find the anti derivative of the following:

Q.1: Rule I

(i) $\int \cos^4 x \sin x\,dx$ (ii) $\int \sqrt{\sin x}\,\cos x\,dx$

(iii) $\int \frac{\sin x\,dx}{\sqrt{1+\cos x}}$ (iv) $\int \sin^2 x \cos^3 x\,dx$

(v) $\int \sin^5 3x\,dx$ (vi) $\int (\tan^4 x + \tan^2 x)\,dx$

(vii) $\int \sqrt{\tan x}\,\sec^4 x\,dx$ (viii) $\int \tan^3 x \sec x\,dx$

(ix) $\int \tan^4 x\,dx$ (x) $\int (x^2+3x+4)^3(2x+3)\,dx$

(xi) $\int \sqrt{x^2+3x+7}\,(2x+3)\,dx$ (xii) $\int \frac{(x^3+x)}{\sqrt[4]{x^4+2x^2+1}}\,dx$

(xiii) $\int \frac{(\ln x)^3}{x}\,dx$ (xiv) $\int \frac{e^x}{(1+e^x)^2}\,dx$

(xv) $\int (\mathrm{Sin}x + \mathrm{Cos}x)^n(\mathrm{Cos}^2 x - \mathrm{Sin}^2 x)\,dx$

Q.2: Rule –II

(i) $\int \frac{dx}{(1+x^2)\tan^{-1} x}$ (ii) $\int \frac{x^3+2}{x^4+8x+10}\,dx$

(iii) $\int \frac{x\,dx}{x^2+1}$ (iv) $\int \frac{dx}{x^{1/3}(x^{2/3}-1)}$

(v) $\int \frac{\sin x}{a+b\cos x}\,dx$ (vi) $\int \frac{\sin 2x}{1+\sin^2 x}\,dx$

(vii) $\int \frac{\sin x + \cos x}{\sin x - \cos x}\,dx$ (viii) $\int \sec x\,dx$

(ix) $\int \mathrm{cosec}\,x\,dx$ (x) $\int \frac{\sin\sqrt{x}}{\sqrt{x}\cos\sqrt{x}}\,dx$

(xi) $\int \frac{\sin x}{4 + 3\,\text{Cos}x}\,dx$ (xii) $\int \frac{e^x + c^{-x}}{e^x - e^{-x}}\,dx$

(xiii) $\int \frac{e^{2x}}{1 + e^{2x}}\,dx$

Answers 7.3

Q.1: Rule – I (i) $-\frac{\cos^5 x}{5} + C$ (ii) $\frac{2}{3}\sin^{3/2} x + C$

(iii) $-2\sqrt{1 + \cos x} + C$ (iv) $\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$

(v) $-\frac{1}{3}\cos 3x + \frac{2}{9}\cos^3 3x - \frac{1}{15}\cos^5 3x + C$

(vi) $\frac{\tan^3 x}{3} + C$ (vii) $\frac{2}{3}\tan^{3/2} x + \frac{2}{7}\tan^{7/2} x + C$

(viii) $\frac{1}{3}\sec^3 x - \sec x + C$ (ix) $\frac{\tan^3 x}{3} - \tan + x + C$

(x) $\frac{1}{4}(x^2 + 3x + 4x)^4 + C$ (xi) $\frac{2}{3}(x^2 + 3x + 7)^{3/2} + C$

(xii) $-\frac{1}{3}(x^4 + 2x^2 + 1)^{3/4} + C$ (xiii) $\frac{1}{4}(\text{In}\, x)^4 + C$

(xiv) $-\frac{1}{1 + e^x} + C$ (xv) $\frac{1}{n + 2}(\text{Sin}x + \text{Cos}x)^{n+2} + C$

Q.2: Rule – II

(i) $\text{In}\tan^{-1} x + C$ (ii) $\frac{1}{4}\text{In}(x^4 + 8x + 10) + C$

(iii) $\frac{1}{2}\text{In}(x^2 + 1) + C$ (iv) $\frac{3}{2}\text{In}(x^{2/3} - 1) + C$

(v) $-\frac{1}{b}\text{In}(a + b\cos x) + C$ (vi) $\text{In}(1 + \text{Sin}^2 x) + C$

(vii) $\text{In}(\sin x - \cos x) + C$ (viii) $\text{In}(\text{Sec}\, x + \tan x) + C$

(ix) $\text{In}(\text{cosec}x - \text{cot}x) + C$ (x) $-2\ln\cos\sqrt{x} + c$

(xi) $-\frac{1}{3}\text{In}(4 + 3\,\text{Cos}x) + C$ (xii) $\text{In}(e^x - e^{-x}) + C$

(xiii) $\frac{1}{2}\text{In}(1 + e^{2x}) + C$

Short Questions

Write the short answers of the following :

Q.1: Find $\int (2x+9)^{-5/2}\,dx$

Q.2: Find $\int \frac{1}{\sqrt[3]{(3x+4)^2}}\,dx$

Q.3: Find $\int \left(x + \frac{1}{x}\right)^2 dx$

S.P **Q.4:** Find $\int \left(\frac{1}{t^3} + \frac{1}{t^2} - 2\right) dt$

Q.5: $\int \frac{-2x}{\sqrt{4-x^2}}\,dx$

Q.6: Find $\int \frac{x^2+1}{x+1}\,dx$

Q.7: Find $\int \left(1 + \frac{3}{x^2}\right)^2 dx$

Q.8: $\int \frac{x}{a+x}\,dx$

Q.9: $\int \frac{x^2}{4+x^2}\,dx$

S.P **Q.10:** Find $\int 8\,(2x+1)^3\,dx$

Q.11. Find $\int \frac{x^3+x^2+x+1}{\sqrt{x}}\,dx$

Q.12: Find $\int \frac{\sin 2x}{\cos^2 2x}\,dx$

Q.13: Find $\int \frac{1}{\sin^2 x \cos^2 x}\,dx$

Q.14: Find $\int \cos x \sin 3x\,dx$

S.P **Q.15:** Evaluate $\int (e^{3x} + e^{5x})\,dx$

Answers

Q1. $-\frac{1}{3}(2x-9)^{-3/2} + c$

Q2. $(3x+4)^{1/3} + c$

Q3. $\frac{x^3}{3} - \frac{1}{x} + 2x + c$

Q4. $-\frac{1}{2t^2} - \frac{1}{t} - 2t + c$

Q5. $2\sqrt{4-x^2} + c$

Q6. $\frac{x^2}{2} - x + 2\ln(x+1) + c$

Q7. $x - \frac{6}{x} - \frac{3}{x^3} + c$

Q8. $x - a\ln(a+x) + c$

Q9. $x - \frac{1}{2}\tan^{-1}\frac{x}{2} + c$

Q10 $(2x+1)^4 + c$

Q11. $\frac{2}{7}x^{7/2} + \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2\sqrt{x} + c$

Q12. $\frac{1}{2}\,\text{Sec}\,2x + c$

Q13. $\tan x - \cot x + c$

Q14. $-\frac{1}{8}[\cos 4x + 2\cos 2x] + c$

Q15. $\frac{1}{3}e^{3x} + \frac{1}{5}e^{5x} + c$

Objective Type questions

Q.1: Encircle the correct one, of the given answers in each item.

1. $\int x^3 dx =$

[✓ (a)] $\frac{x^4}{4}$ (b) $\frac{x^4}{3}$

(c) $3x^2$ (d) $4x^4$

2. $\int e^{2x} dx =$

[✓ (a)] $\frac{e^{2x}}{2}$ (b) $\frac{e^{x^2}}{2}$

(c) $2e^{2x}$ (d) $\frac{e^{2x+1}}{2}$

3. $\int \frac{\cos x}{\sin x} dx =$

(a) $\ell n \cos x$ [✓ (b)] $\ell n \sin x$

(c) $\ell n \cot x$ (d) $\frac{\cos^2 x}{2}$

4. $\int \tan x \sec^2 x \, dx =$

(a) $\ell n \tan x$ [✓ (b)] $\frac{\tan^2 x}{2}$

(c) $\frac{\sec^2 x}{3}$ (d) $\sec x \tan x$

5. $\int x^{n+1} dx =$

(a) $\frac{x^{n+1}}{n+2}$ [✓ (b)] $\frac{x^{n+2}}{n+2}$

(c) $(n+1)x^n$ (d) $\frac{x^2}{n}$

6. $\int \sqrt{x} \, dx =$

(a) $\frac{1}{2}x^{1/2}$ (b) $\frac{2x^{1/2}}{3}$

(c) $\frac{2}{3}x^{3/2}$ (d) $\frac{1}{2x^{\frac{1}{2}}}$

7. $\int \sec x \, dx$

(a) $\tan x$ (b) $\frac{\sec^2 x}{2}$

(c) $\ln(\sec x + \tan x)$ (d) $\sec x \tan x$

8. $\int (ax + b)\, dx =$

(a) $\frac{(ax + b)^2}{2a}$ (b) $\frac{(ax + b)^2}{2}$

(c) $\ln(ax + b)$ (d) $a(ax + b)$

9. $\int \frac{a + x}{x}\, dx =$

(a) $a \ln x + x$ (b) $\frac{(ax + b)^2}{2}$

(c) $\ln x + a$ (d) $x + a$

10. $\int \frac{\operatorname{cosec}^2 x}{\cot x}\, dx =$

(a) $-\ln \cot x$ (b) $\ln \cot x$

(c) $\frac{\cot^2 x}{2}$ (d) $\ln(\operatorname{cosec}^2 x)$

Answers

Q.1:

1.	a	**2.**	a	**3.**	b	**4.**	b	**5.**	b
6.	c	**7.**	c	**8.**	a	**9.**	a	**10.**	a

Chapter 8

Methods of Integration

8.1 Introduction

In the previous chapter we have been dealing with the integration of the functions whose derivatives are easily recognized as a part of integrand. But there are some other cases in which it is very difficult to recognize them. In such cases we are using some other methods. These methods are explained as under:

1. Integration by Substitution
2. Integration by Parts

8.2 Integration by Substitution:

Sometimes the change of the variable by means of a substitution changes the integrand either in standard form or a form easier to deal with. After evaluating the new integral, we have to replace back with its original variable.

In fact there is no general rule for proper substitution, but following tips will prove very helpful.

Type 1:

If a part of the integrand is such that its differential coefficient is a factor of the integrand. Then put that part equal to t i.e., to integrate the function of the type $f(x^n).\, x^{n-1}$, always put $x^n = t$ its differential is $nx^{n-1}\, dx = dt$.

Example 1:

(i) $\int x^2 \cos x^3 dx$ (ii) $\int 2x e^{x^2} dx$

Solution:

In this case differential coefficient of x^3 is $3x^2$ which is a factor of integrand i.e $x^2 \cos x^3$.

$\therefore$ Put $x^3 = t$

$3x^2\, dx = dt$

$$\therefore \quad I = \int x^2 \cos t \frac{dt}{3x^2}$$

$$= \frac{1}{3}\int \cos t\, dt \qquad = \frac{1}{3}\sin t + c$$

$$= \frac{1}{3}\sin x^2 + c \qquad \text{Ans.}$$

(ii) put $u = x^2$, then $du = 2x\,dx$

Now , $I = \int 2x e^{x^2} = \int e^{x^2} 2x\,dx$

$$= \int e^u\,du = e^u + c$$

$$= e^{x^2} + c$$

Type 2:

If the integrand contains In x. it may be helpful to put In x = t

Example 2: $I = \int \frac{dx}{x(\ln x)}$

Solution: $I = \int \frac{1}{\ln x} \cdot \frac{1}{x}\,dx$

Put In x = t

$\frac{1}{x}\,dx = dt$

$I = \int \frac{1}{t} \cdot dt = \ln|t| + c$

$= \ln|\ln x| + c$ Ans.

Type 3:

If the integrand contains an inverse trigonometric function of x ($\sin^{-1}x$. etc). Put the function equal to t.

Example 3: $I = \int \frac{\sin^{-1}x}{\sqrt{1 - x^2}}\,dx$

Solution:

Put $\sin^{-1}x = t$

$x = \sin t$

$dx = \cos t\,dt$

$I = \int \frac{t}{\sqrt{1 - \sin^2 t}} \cos t \cdot dt$

$= \int \frac{t}{\cos t} \cos t \cdot dt$

$= \int t\,dt$

$= \frac{t^2}{2} + c$

$$= \frac{\left(\sin^{-1} x\right)^2}{2} + c \qquad \text{Ans.}$$

Type 4:

To integrate odd power of cos x
Put sin x = t

Example 4: $I = \int \cos^3 x \, dx$

Solution:

$$= \int \cos^2 x \cos x \, dx$$

$$= \int (1 - \sin^2 x) \cos x \, dx$$

Put $\sin x = t \Rightarrow \cos x \, dx = dt$

$$I = \int (1 - t^2) \, dt$$

$$= \int dt - \int t^2 \, dt$$

$$= t - \frac{t^2}{3} + c$$

$$= \sin x - \frac{\sin^3 x}{3} + c$$

Type 5:

To integrate odd power of sin x .

Put cos x = t

Example 5: $I = \int \sin^5 x \, dx$

Solution:

$$I = \int \sin^4 x \sin x \, dx$$

$$= \int (\sin^2 x)^2 \sin x \, dx$$

$$= \int (1 - \cos^2 x)^2 \sin x \, dx$$

Put $\cos x = t$

$-\sin x . dx = dt$

$$dx = \frac{dt}{-\sin x}$$

$$\therefore \quad I = \int (1 - t^2)^2 \sin x . \frac{dt}{-\sin x}$$

$$= - \int (1 + t^4 - 2t^2)\, dt$$

$$= - \int dt - \int t^4 \, dt + 2 \int t^2 \, dt$$

$$= -t - \frac{t^5}{5} + 2\frac{t^3}{3} + c$$

$$= - \cos x - \frac{\cos^5 x}{5} + \frac{2}{3} \cos^3 x + c \qquad \text{Ans.}$$

Type 6:

To integrate the product $(\sin x : \cos x)$ put $\cos x = t$, if the power of $\cos x$ is even and make either substitution if the power of the both are odd. In case the powers of both are even, neither of the substitution may be useful.

Example 6: $I = \int \cos^4 x . \sin x \, dx$

Solution:

$\therefore$ Sin x contains odd power

Put $\cos x = t$

$-\sin x \, dx = dt$

$$dx = \frac{dt}{-\sin x}$$

$$\therefore \quad I = \int t^4 \sin x . \frac{dt}{-\sin x}$$

$$= -\int t^4 . dt$$

$$= -\frac{t^5}{5} + c$$

$$= -\frac{\cos^5 x}{5} + c \qquad \text{Ans.}$$

Example 7:

$$1 = \int \sin^6 x \cos x.\, dx$$

Solution:

$\therefore$ cos x contains odd power

Put $\sin x = t$

$\text{Cos } x \; dx = dt$

$$dx = \frac{dt}{\cos x}$$

$$\therefore \quad 1 = \int t^6 .\cos x. \frac{dt}{\cos x}$$

$$= \int t^6 \, dt$$

$$= \frac{t^7}{7} + c$$

$$= \frac{\sin^7 x}{7} + c$$

Exercise 8.1

Q.1: Evaluate the following integrates [Type – 1]

(i) $\int (ax^2 + 2bx + c)^n \,(ax + b)\, dx$ (ii) $\int 3x\sqrt{1 - 2x^2}\, dx$

(iii) $\int x\sqrt{x - a}\, dx$ (iv) $\int \frac{x + 2}{\sqrt{2x^2 + 8x + 9}}\, dx$

(v) $\int \frac{1}{\sqrt{x}} \sin\sqrt{x}\, dx$ (vi) $\int e^{\tan x} \sec^2 x.\, dx$

(vii) $\int \frac{\sec^2 x.\, dx}{\sqrt{3\tan x + 1}}$ (viii) $\int x.e^{x^2}\, dx$

(ix) $\int \frac{e^{4x}}{1 + 3e^{4x}}\, dx$ (x) $\int e^x \sin e^x\, dx$

Q.2: **Integrate the following [Type – 2]**

S.P

(i) $\int \frac{\ell n\, x}{x}\, dx$ (ii) $\int \frac{dx}{x\,(1 + \ln x)}$

(iii) $\int \frac{\cot x}{\ln \sin x}\, dx$ (iv) $\int \frac{\sec x \operatorname{cosec} x}{\ln \tan x}\, dx$

(v) $\int \frac{\tan(\ell n x)}{x}\, dx$

Q.3: **Integrate the following [Type – 3]**

(i) $\int \frac{\cos^{-1} x}{\sqrt{1 - x^2}}\, dx$ (ii) $\int \frac{e^{m\tan^{-1} x}}{1 + x^2}\, dx$

(iii) $\frac{(\tan^{-1} x)^3}{1 + x^2}\, dx$

Q.4: **Evaluate the following integrate [Type – 4][Type – 5][Type – 6]**

(i) $\int \sin^3 x\, dx$ (ii) $\int \cos^5 x\, dx$

(iii) $\int \sin^5 x.\, \cos x\, .\, dx$ (iv) $\int \sin^2 x \cos^3 x\, dx$

(v) $\int \tan x \sec^3 x\, dx$ (vi) $\int \cot^3 2x \operatorname{cosec}^3 2x\, dx$

Answers 8.1

Q.1: (i) $\frac{\left(ax^2 + 2bx + c\right)^{n+1}}{2(n+1)}$ (ii) $-\frac{1}{2}\,(1 - 2x^2)^{3/2} + c$

(iii) $(x - a)^{3/2}\left[\frac{2a}{3} + \frac{2}{5}(x - a)\right]$ (iv) $\frac{1}{2}\sqrt{2x^2 + 8x + 9} + c$

(v) $-2 \cos \sqrt{x} + c$ (vi) $e^{\tan x} + c$

(vii) $^2/3\,\sqrt{2\tan x + a} + c$ (viii) $\frac{1}{2} e^{x^2} + c$

(ix) $\frac{1}{12}\, \ell n\,(1 + 3e^{4x}) + c$ (x) $-\cos e^x + c$

Q.2: (i) $\frac{1}{2}(\ln x)^2 + c$ (ii) $\ln(1 + \ln x) + c$

(iii) $\ln(\ln \sin x) + c$ (iv) $\ln(\ln \tan x) + c$

(v) $\ell n \sec(\ell n x) + c$

Q.3: (i) $-\frac{1}{2}(\cos^{-1}x)^2+c$ (ii) $\frac{1}{m}e^{m\tan^{-1}x}+c$

(iii) $\frac{1}{4}(\tan^{-1}x)^4+c$

Q.4: (i) $-\cos x+\frac{\cos^3 x}{3}+c$ (ii) $\sin x-\frac{2}{3}\sin^3 x+\frac{1}{5}\sin^5 x+c$

(iii) $\frac{1}{6}\sin^6 x+c$ (iv) $\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+c$

(v) $\frac{1}{3}\sec^3 x+c$

(vi) $-\frac{1}{2}\left[\frac{1}{5}\operatorname{cosec}^5 2x-\frac{1}{3}\operatorname{cosec}^3 2x\right]+c$

8.3 Trigonometric substitutions

A change of variable involving Trigonometric functions is called a **Trigonometric substitutions.** This type of substitution is particularly useful when an integral is a rational function of x and one of the following radical expressions. For each of these radical we use the following substitutions

Radical		**SUBSTITUTION**
$\sqrt{ax+b}$	put	$ax+b=t^2$
$\sqrt{x+a}$	put	$x+a=t^2$
$\sqrt{a^2-x^2}$	put	$x=a\sin t$
$\sqrt{a^2+x^2}$	put	$x=a\tan t$
$\sqrt{x^2-a^2}$	put	$x=a\sec t$

8.4 Integrals giving inverse trigonometric functions

The integration formulas are

1. (a) $\int\frac{dx}{1+x^2}=\tan^{-1}x+c$ or $-\cot^{-1}x+c$

(b) $\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$ or $-\frac{1}{a}\cot^{-1}\frac{x}{a}+c$

2. (a) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c$ or $-\cos^{-1}x + c$

(b) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$ or $-\cos^{-1}\frac{x}{a} + c$

3. (a) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + c$ or $-\operatorname{cosec}^{-1}x + c$

(b) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$ or $-\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a} + c$

These can be easily verified by differentiating the right hand members.

Note: That **$\sin^{-1}x$ = arc sin x.** similar relation is used for others

Example 1: $I = \int \frac{dx}{\sqrt{a^2-x^2}}$ (Proof of formula 2b)

Solution:

Put $x = a\sin t$

$dx = a\cos t\,dt$

$$I = \int \frac{a\cos t\,dt}{\sqrt{a^2-a^2\sin^2 t}} = \int \frac{a\cos t\,dt}{a\sqrt{1-\sin^2 t}}$$

$$= \int \frac{\cos t}{\cos t}dt$$

$$= \int dt$$

$$= t + c$$

$$= \sin^{-1}\frac{x}{a} + c$$ Ans.

Example 2: $\int \frac{dx}{4+x^2}$

Solution:

$$I = \int \frac{dx}{4+x^2}$$

$$= \int \frac{dx}{(2)^2 + (x)^2}$$

Put $x = 2\tan t$

$dx = 2\sec^2 t\,dt$

$$= \int \frac{2}{4 + (2\tan t)^2}\sec^2 t\,dt$$

$$= \int \frac{2\sec^2 t\,dt}{4 + 4\tan^2 t}$$

$$= \frac{1}{2}\int \frac{\sec^2 t}{\sec^2 t}\,dt \qquad \text{as} \qquad 1 + \tan^2 t = \sec^2 t$$

$$= \frac{1}{2}\int dt \qquad , x = 2\tan t$$

$$= \frac{1}{2}t + c \qquad , \frac{x}{2} = \tan t \Rightarrow t = \tan^{-1} x/2$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c \qquad \text{Ans.}$$

Example 3: $\int \frac{dx}{(a^2 + x^2)^{3/2}}$

Solution:

Put $x = a\tan t$

$dx = a\sec^2 t\,dt$

$$\int \frac{a\sec^2 t\,dt}{(a^2 + a^2\tan^2 t)^{3/2}}$$

$$= \int \frac{a\sec^2 t\,dt}{(a^2)^{3/2}(1 + \tan^2 t)^{3/2}}$$

$\tan t = \frac{x}{a}$

$\sin t = \frac{x}{\sqrt{x^2 + a^2}}$

$$= \int \frac{a\sec^2 t\,dt}{a^3(\sec^2 t)^{3/2}} = \frac{1}{a^2}\int \frac{\sec^2 t\,dt}{\sec^3 t}$$

$$= \frac{1}{a^2}\int \frac{dt}{\sec t} = \frac{1}{a^2}\int \cos t.\,dt$$

$$=\frac{1}{a^2}\ \text{Sin}\, t+c$$

$$=\frac{x}{a^2\sqrt{a^2+x^2}}+c \qquad \text{Ans.}$$

Exercise 8.2

Q.1:

(i) $\int\frac{dx}{\sqrt{25-16x^2}}$ (ii) $\int\sqrt{a^2-x^2}\,dx$

(iii) $\int\frac{1}{a^2-x^2}\,dx$ (iv) $\int\frac{dx}{\sqrt{a^2+x^2}}$

(v) $\int\frac{x^2}{a^2+x^2}\,dx$ (vi) $\int\frac{1}{a^2+9x^2}\,dx$

(vii) $\int\frac{dx}{16+9x^2}$ (viii) $\int\frac{dx}{x\sqrt{9x^2-4}}$

(ix) $\int\frac{dx}{(x^2+a^2)^2}$ (x) $\int\frac{dx}{(a^2-x^2)^{3/2}}$

Answers 8.2

Q.1:

(i) $\frac{1}{4}\sin^{-1}\frac{4x}{5}+c$ (ii) $\frac{a^2}{2}\sin^{-1}x/a+\frac{x\sqrt{a^2-x^2}}{2}+c$

(iii) $\frac{1}{a}\ln\left[\frac{a+x}{\sqrt{a^2-x^2}}\right]+c$ or $\frac{1}{2a}\,\text{In}\,\frac{a+x}{a-x}+c$ (iv) $\text{In}\left(\frac{x+\sqrt{x^2+a^2}}{a}\right)+c$

(v) $x-a\tan^{-1}x/a+c$ (vi) $\frac{1}{3a}\tan^{-1}\left(\frac{3x}{a}\right)+c$

(vii) $\frac{1}{12}\tan^{-1}\left(\frac{3x}{4}\right)+c$ (viii) $\frac{1}{2}\sec^{-1}\frac{3x}{2}+c$

(ix) $\frac{1}{2a^3}\left[\tan^{-1}\left(\frac{x}{a}\right)+\frac{ax}{a^2+x^2}\right]+c$ (x) $\frac{1}{a^2}\,\frac{x}{\sqrt{a^2-x^2}}+c$

8.5 Integration by Parts:

When an expression involving product of logarithms , inverse functions or complicated integrals cannot be evaluated directly by the use of standard forms, one of the most useful techniques for transforming it to a standard form is the formula for integration by parts. This formula is based on the inverse of the formula for differentiation of a product.

If u and v are functions of a single independent variable x, then from the formula for the differentiation of a product.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(I)$$

Let: $\frac{dv}{dx} = w \quad \Rightarrow \quad dv = w\,dx$

Integrating both sides , we get

$v = \int w\,dx$

Equation (1) becomes

$$\frac{d}{dx}(uv) = uw + \frac{du}{dx}\int w\,dx$$

Integration both sides w.r.t. x. we get.

$$uv = \int u\,w\,dx + \int\left(\frac{du}{dx}\int w.dx\right)dx$$

Or $$\int uw\,dx = u\int w\,dx - \int\left(\frac{du}{dx}\int w.dx\right)dx$$

Before applying formula we will have to specify the first and the second function. In formula **u** is the first and **w** is the second function.

Example:

$\int x.\cos x.\,dx$

Here 1^{st} function = x , 2^{nd} function = cos x

$$\int x.\cos x.\,dx = x\int \cos x\,dx - \int[\frac{d}{dx}(x)\int \cos x\,dx]\,dx$$

$$= x.\sin x - \int 1.\sin x + c$$

$$= x.\sin x + \cos x + c$$

Note:

(i) Selection of the first and second function should be made in such a way that first function becomes simple on differentiation and the second function on integration.

(ii) For the integration of a simple function we take the integrand as the first function and the unity as the second function. This method is especially useful in the case of logarithmic and inverse trigonometric functions.

(iii) The inverse trigonometric or logarithmic functions are to be taken as 1st function always.

Repeated Integration by Parts:

Sometimes it is required to repeat the method of integration by parts to integrate a function completely.

Illustrative Examples

Evaluate the following integrals :

(i) $\int x.\sec^2 x.\,dx$ **(ii)** $\int x\tan^{-1} x\,dx$ **(iii)** $\int \text{In}\,x.\,dx$

(iv) $\int x^2.\,e^x\,dx$ **(v)** $\int e^{ax}\sin bx\,dx$ **(vi)** $\int\sqrt{a^2-x^2}\,dx$

(vi) $\int e^x(\sin x+\cos x)\,dx$

Solution (i) $\int x.\sec^2 x.\,dx$

Here first function = x, second = $\sec^2 x$

$$\int x.\sec^2 x.\,dx = x.\tan x - \int 1.\tan x,\,dx$$

$$= x\tan x - \int \tan x.\,dx$$

$$= x\tan x - \text{In}\sec x + c \qquad \text{Ans.}$$

(ii) $\int x\tan^{-1} x\,dx$

Solution:

Take first function = $\tan^{-1} x$, Second function = x

$$\int x.\tan^{-1} x\,dx = \tan^{-1} x\,\frac{x^2}{2} - \frac{1}{2}\int\frac{x^2}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int\frac{x^2}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int\left[1-\frac{1}{1+x^2}\right]dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{dx}{1+x^2}$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + c$$

$$= \frac{x}{2}(x^2+1)\tan^{-1}x - \frac{x}{2} + c \qquad \text{Ans.}$$

(iii) $\int ln\,x.\,dx$

Solution:

Take *ln* x = first function, 1 = second function.

$$\int ln\,x\,dx = ln\,x.\,x - \int \frac{1}{x}x\,dx$$

$$= x.\,ln\,x - \int dx$$

$$= x.\,ln\,x - x + c$$

$$= x(\,ln\,x - 1) + c \qquad \text{Ans.}$$

(iv) $\int x^2.\,e^x\,dx$

Solution:

Take first function x² and second functions as eˣ.

$$\int x^2.\,e^x\,dx\,dx = x^2e^x - \int 2xe^x\,dx$$

$$= x^2.\,e^x - 2\int x\,e^x\,dx$$

$$= x^2.\,e^x - 2\,[x\,e^x - \int e^x\,dx]$$

$$= x^2.\,e^x - 2x\,e^x + 2\,e^x + c$$

$$= e^x(x^2 - 2x + 2) + c \qquad \text{Ans.}$$

(v) $\int e^{ax}\sin bx\,dx$

Solution:

Let $1 = \int e^{ax}\sin bx\,dx$

Take e^{ax} as first function and sin bx as second function.

$$I = e^{ax}\left(\frac{-\cos bx}{b}\right) - \int\left(\frac{-\text{cons}bx}{b}\right) ae^{ax}\,dx$$

$$I = -e^{ax}\frac{\cos bx}{b} + \frac{a}{b}\int e^{ax}.\cos bx.dx$$

$$I = -e^{ax}\frac{\cos bx}{b} + \frac{a}{b}\left[e^{ax}\frac{\sin bx}{b} - \int\frac{\sin bx}{b}\,a.e^{ax}.dx\right]$$

$$I = -e^{ax}\frac{\cos bx}{b} + \frac{a}{b}e^{ax}\sin bx - \frac{a^2}{b^2}I$$

$$I + \frac{a^2}{b^2}I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2}e^{ax}\sin bx$$

$$\left(1 + \frac{a^2}{b^2}\right)I = \frac{e^{ax}}{b^2}[-b\cos bx + a\sin bx]$$

$$\left(\frac{a^2 + b^2}{b^2}\right)I = \frac{e^{ax}}{b^2}[a\sin bx - b\cos bx]$$

$$I = \frac{b^2}{(a^2 + b^2)}\frac{e^{ax}}{b^2}[a\sin bx - b\cos bx]$$

$$I = \frac{e^{ax}}{a^2 + b^2}[a\sin bx - b\cos bx] + c \qquad \text{Ans.}$$

(vi) $\int\sqrt{a^2 - x^2}\,dx$

Solution:

Here take $\sqrt{a^2 - x^2}$ as first function and 1 as second function.

Let $I = \int\sqrt{a^2 - x^2}\,dx$

$\therefore \quad I = \int\sqrt{a^2 - x^2}.1\,dx$

$$I = \sqrt{a^2 - x^2}\;x - \int x\frac{1}{2}(a^2 - x^2)^{-1/2}(-2x)\,dx$$

$$I = x\sqrt{a^2 - x^2} - \int(a^2 - x^2)^{-1/2}(-x^2)\,dx$$

$$I = x\sqrt{a^2 - x^2} - \int\frac{-x^2}{\sqrt{a^2 - x^2}}\,dx$$

$$I \quad = x\sqrt{a^2-x^2} - \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$I \quad = x\sqrt{a^2-x^2} - \int \frac{a^2-x^2}{\sqrt{a^2-x^2}}\,dx + a^2\int \frac{dx}{\sqrt{a^2-x^2}}$$

$$I \quad = x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\,dx + a^2\int \frac{dx}{\sqrt{a^2-x^2}}$$

$$I \quad = x\sqrt{a^2-x^2} - I + a^2\int \frac{dx}{\sqrt{a^2-x^2}}$$

$$21 \quad = x\sqrt{a^2-x^2} + a^2 \sin^{-1}\frac{x}{a}$$

$$I \quad = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

$$\therefore \quad \int\sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

(vi) $\int e^x(\sin x + \cos x)\,dx$

Solution :

$$\int e^x(\sin x + \cos x)\,dx = \int e^x \sin x\,dx + \int e^x \cos x\,dx$$

Take sin x as first and e^x as second functions.

$$= \sin x\, e^x - \int \cos x\, e^x\,dx + \int e^x \cos x\,dx.$$

$$= e^x \sin x + c$$

Exercise 8.3

Q.1: Evaluate the following integrals.

(i) $\int x^2 \sin x\,.\,dx$ (ii) $\int x \cos 3x\,.\,dx$ (iii) $\int x.\cos^2 x\,dx$

(iv) $\int x \sin x \cos x\,.\,dx$ (v $\int \sec^3 x.\,dx$ (vi) $\int \frac{x}{\sin^2 x}\,dx$

(vii) $\int x\,.\,\sin x\,dx$

Q.2: Integrate the following.

(i) $\int x \sin^{-1} x \, . \, dx$ (ii) $\int x^2 \tan^{-1} x \, . \, dx$

(iii) $\int x^3 . \tan^{-1} x \, dx$ (iv) $\int \sin^{-1} x . \, dx$

(v) $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \, dx$ (vi) $\int \frac{x^2 \tan^{-1} x}{1+x^2} \, dx$

(vii) $\int x \sec^{-1} x . \, dx$

Q.3: Evaluate the following integrals.

(i) $\int x \, \text{In} \, x \, . \, dx$ (ii) $\int x^2 \, \text{In} \, x \, . \, dx$

(iii) $\int \ell n \, (x^2+1) dx$ (iv) $\int (\text{In} \, x)^2 \, dx$

(v) $\int \text{In} \left(x + \sqrt{x^2+1} \right) dx$ (vi) $\int \sin x \, \text{In} \cos x . \, dx$

(vii) $\int \sec^2 x . \, \text{In} \tan x . \, dx$

Q.4: Integrate the following.

(i) $\int x^3 e^{2x} dx$ (ii) $\int x^3 e^{-x} dx$

(iii) $\int 2x^3 e^{x^2} dx$ (iv) $\int x^2 e^{ax} dx$

Q.5: Evaluate the following integrals.

(i) $\int e^x \sin x$ (ii) $\int e^x \sin x \cos x \, dx$

(iii) $\int e^{3x} \cos x \, dx$ (iv) $\int e^{3x} \sin 2x \, dx$

(v) $\int e^{ax} \cos bx . \, dx$

Q.6: Integrate the following:

(i) $\int \sqrt{a^2+x^2} \, dx$ (ii) $\int \sqrt{x^2-a^2} \, dx$

(iii) $\int \sqrt{4-x^2} \, dx$ (iv) $\int \sqrt{25-x^2} \, dx$

Answers 8.3

Q.1: (i) $2x\sin x + (2 - x^2)\cos x + c$ (ii) $\frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x + c$

(iii) $\frac{x^2}{4} + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + c$ (iv) $\frac{\sin 2x}{8} - \frac{x\cos 2x}{4} + c$

(v) $\frac{1}{2}[\sec x\tan x + \mathrm{In}(\sec x + \tan x)] + c$

(vi) $\ell n \sin x - x\cot x + c$ (vii) $-x\cos x + \sin x + c$

Q.2: (i) $\frac{x^2}{2}\sin^{-1}x - \frac{1}{4}\sin^{-1}x + \frac{x}{4}\sqrt{1-x^2} + c$

(ii) $\frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + \frac{1}{6}\mathrm{In}(1+x^2) + c$

(iii) $\frac{1}{4}(x^4 - 1)\tan^{-1}x - \frac{1}{12}x^3 + \frac{1}{4}x + c$

(iv) $x\sin^{-1}x + \sqrt{1-x^2} + c$ (v) $-\sin^{-1}x\sqrt{1-x^2} + x + c$

(vi) $x\tan^{-1}x - \frac{1}{2}\mathrm{In}(1+x^2) - \frac{1}{2}(\tan^{-1}x)^2 + c$

(vii) $\frac{x^2}{2}\sec^{-1}x - \frac{1}{2}\sqrt{x^2-1} + c$

Q.3: (i) $\frac{x^2}{2}\mathrm{In}\,x - \frac{1}{4}x^2 + c$ (ii) $\frac{x^3}{3}\mathrm{In}\,x - \frac{x^3}{9} + c$

(iii) $x\,\ell n(x^2+1) - 2x + 2\tan^{-1}x + c$

(iv) $x(\mathrm{In}\,x)^2 - 2x\,\mathrm{In}\,x + 2x + c$

(v) $x\,\mathrm{In}(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + c$

(vi) $\cos x(1 - \mathrm{In}\cos x) + c$

(vii) $\tan x(\mathrm{In}\tan x - 1) + c$

Q.4: (i) $e^{2x}\left[\frac{x^3}{2}-\frac{3x^2}{4}+\frac{3x}{4}-\frac{3}{8}\right]+c$

(ii) $-(x^3+3x^2+6x+6)\,e^{-x}+c$

(iii) $e^{x^2}(x^2-1)+c$ (iv) $\frac{e^{ax}}{a}\left(x^2-\frac{2x}{a}+\frac{2}{a^2}\right)+c$

Q.5: (i) $\frac{e^x}{2}(\sin x-\cos x)+c$ (ii) $\frac{e^x}{5}\left[\frac{1}{2}\sin 2x-\cos 2x\right]+c$

(iii) $\frac{1}{10}e^{3x}(\sin x+3\cos x)+c$

(iv) $\frac{e^{3x}}{13}(3\sin 2x-2\cos 2x)+c$

(v) $\frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)+c$

Or $\frac{e^{ax}}{a^2+b^2}(b\sin bx+a\cos bx)+c$

Q.6: (i) $\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\ln\left[x+\frac{a^2+x^2}{a}\right]+c$

(ii) $\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\ln\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)+c$

(iii) $x\frac{\sqrt{4-x^2}}{2}+2\sin^{-1}\frac{x}{2}+c$

(iv) $\frac{x\sqrt{25-x^2}}{2}+\frac{25}{2}\sin^{-1}\frac{x}{2}+c$

Short Questions

Write the short answers of the following

Q.1: Find the value of $\int 10\,(x^2 - 3x + 4)^9\,(2x - 3)\,dx$

Q.2: Find $\int x^4 \sec^2 (x^5).\,dx$,

Q:.3 Find $6\int x^2 . e^{x^3}\,dx$,

Q.4: Find the value of $\int \frac{\sin^{-1}x}{\sqrt{1 - x^2}}\,dx$

Q.5: Find $\int \frac{\cos(\ln x)}{x}\,dx$

Q.6: Find the value of $\int \frac{x - 1}{x^2 - 2x + 3}\,dx$

Q.7: Find the value of $\int \cot^2 x\,dx$.

Q.8: Find $\int 2\sec 2x\,dx$

Q.9: Find $\int \sin^9 x \cos x\,dx$

Q.10: Find $\int \operatorname{cosec}^2 x \sqrt{\cot x}\,dx$

Q.11: Find the value of $\int x\,e^x\,dx$

Q.12: Find the value of $\int \tan^{-1}x\,dx$.

Q.13: Find $\int e^x(\sin x + \cos x)\,du$

Q.14: $\int \frac{\cos x}{\sqrt{\sin x}}\,dx$

Q.15: $\int \frac{\sin^2 x}{\cos^4 x}\,dx$

Q.16: $\int \frac{1}{25 + x^2}\,dx$

Answers

Q1. $(x^2 - 3x + 4)^{10} + c$ **Q2.** $\frac{1}{5}\tan x^5 + c$ **Q3.** $2e^{x^3} + c$

Q4. $\frac{1}{2}(\text{Sin}^{-1}x)^2 + c$ **Q5.** $\sin(\ln x) + C$ **Q6.** $\frac{1}{2}\ell n(x^2 - 2x + 3) + C$

Q7. $-\cot x - x + c$ **Q8.** $\ell n(\text{Sec}\,2x + \tan 2x) + c$ **Q9.** $\frac{1}{10}\sin^{10}x + c$

Q10. $\frac{2}{3}\cot^{3/2}x + c$ **Q11.** $e^x(x - 1) + c$

Q12. $x\tan^{-1}x - \frac{1}{2}\ell n(1 + x^2) + c$ **Q13.** $e^x \sin x + c$ **Q14.** $2\sqrt{\text{Sin}\,x}$

Q15 $\frac{1}{3}\tan^3 x$ **Q16.** $\frac{1}{5}\tan^{-1}\frac{x}{5}$

Objective Type questions

Q. 1: Encircle the correct one, of the given answers in each item.

1. $\int \frac{1}{1-x^2}\,dx =$

[✓ (a) $\text{Sin}^{-1} x$]
(b) $\text{Cos}^{-1} x$
(c) $\text{Sec}^{-1} x$
(d) $\tan^{-1} x$

S.P.

2. $\int \frac{-1}{\sqrt{1-x^2}}\,dx =$

(a) $\text{Sin}^{-1} x$
[✓ (b) $\text{Cos}^{-1} x$]
(c) $\text{Sec}^{-1} x$
(d) $\sqrt{1-x^2}$

3. $\int \frac{e^x}{1+e^x}\,dx =$

(a) $1+e^x$
[✓ (b) $\ell n\,(1+e^x)$]
(c) e^x
(d) $\frac{(1+e^x)^2}{2}$

4. $\int \frac{\text{Sec}\,x \tan x}{3+\text{Sec}\,x}\,dx =$

(a) $\text{Sec}x + \tan x$
(b) $3 + \sec x$
(c) $\ln\,(3+\sec x)$
(d) $\ln(\sec x + \tan x)$

5. $\int x \sin x\,dx =$

[✓ (a) $-x\,\text{Cos}\,x + \text{Sin}\,x$]
(b) $\text{Sin}\,x$
[✓ (c) $x + \text{Sin}\,x$]
(d) $\frac{x^2}{2}\cos x$

6. $\int \text{Sin}^4 x\,\text{Cos}x\,dx =$

[✓ (a) $\frac{\text{Sin}^5 x}{5}$]
(b) $\frac{\text{Sin}^5 x\,\text{Cos}\,x}{5}$

(c) $\frac{\text{Cos}^2 x}{2}$ (d) $-\sin x \cos x$

7. $\int x \,\text{Sec}^2 x \, dx =$

(a) $x \tan x$ [✓ (b) $x \tan x + \ell n \sec x$]

(c) $\text{Tan}x$ (d) $x \tan x - \ell n \sec x$

8. $\int xe^x \, dx =$

(a) $x e^x + e^x$ [✓ (b) $x e^x - e^x$]

(c) e^x (d) $\frac{x^2}{2} e^x$

9. $\int (ax + b)^3 \, dx =$

[✓ (a) $\frac{(ax+b)^4}{4a}$] (b) $3(ax + b)^2$

(c) $\frac{(ax+b)^3}{4a}$ (d) $3 a (ax + b)^2$

10. $\int \text{Cosec}\, x \, dx =$

[✓ (a) In (Cosec x – cosec x)] (b) $\ell n \sec x$

(c) In(Cosecx + cotx) (d) cosx

Answers

Q.1:

1	a	2	b	3	b	4	c	5	a
6	a	7	b	8	b	9	a	10	a

Chapter 9

Definite Integral and Applications

9.1 Definite Integral

Let f(x) be a continuous function in an interval $a \le x \le b$. If $\int f(x)\,dx = g(x)$, then g(b) – g(a) is called the definite integral of f(x) between the limits a and b. It is denoted by $\int_a^b f(x)\,dx$

(read as integral of f(x) from a to b)

Here "a" is called the inferior or lower limit
"b" is called superior or upper limit
The interval [a, b] is called the range of integration.
The difference g(b) – g(a) is denoted by $[g(x)]_a^b$

Thus $\int_a^b f(x)\,dx = [g(x)]_a^b = g(b) - g(a)$

Note: The definite integral is so called because the arbitrary constant of indefinite integral disappears from it.

9.2 Properties of Definite Integrals:

If f(x) and g(x) are two continuous functions in the interval $a \le x \le b$ then

1. If upper and lower limits are same, then value of integral is zero i.e

$$\int_a^a f(x)\,dx = 0$$

2. Change of variable does not affect the value of integral i.e.,

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

3. Interchange of limits, change the sign of value of the Integral i.e.,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

4. $$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$$

5. $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Where c is any constant within or outside the interval [a, b]

6. Sum and difference $\int_a^b [f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)dx$

Example 1: Evaluate

(i) $\int_1^3 x^2\,dx =$ (ii) $\int_0^2 \frac{x^3}{x+1}dx$

Solution:

(i) $$\int_1^3 x^2\,dx = \left[\frac{x^3}{3}\right]_1^3$$

$$= \frac{1}{3}[3^3 - 1^3]$$

$$= \frac{1}{3}[27 - 1]$$

$$= \frac{26}{3}\text{ Ans}$$

(ii) $$\int_0^2 \frac{x^3}{x+1}dx$$

$$= \int_0^2 \left[x^2 - x + 1 - \frac{1}{x+1}\right]dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} + x - \text{In}(x+1)\right]_0^2$$

$$= \left[\frac{8}{3} - \frac{4}{2} + 2 - \text{In}(2+1)\right] - [0]$$

$$= \frac{8}{3} - 2 + 2 - \text{In}\,3$$

$$= \frac{8}{3} - \text{In}\,3 \text{ Ans}$$

Example 2: Evaluate

(i) $\int_{-1}^{1} (3x^2 - x^3)dx$ (ii) $\int_{2}^{3} \frac{x.dx}{1+x^2}$

Solution:

(i) $\int_{-1}^{1} (3x^2 - x^3)dx$

$$= \left[\frac{3x^3}{3} - \frac{x^4}{4}\right]_{-1}^{1} = \left[x^3 - \frac{x^4}{4}\right]_{-1}^{1}$$

$$= \left[\left((1)^3 - \frac{(1)^4}{4}\right) - \left((-1)^3 - \frac{(1)^4}{4}\right)\right]$$

$$= 1 - \frac{1}{4} + 1 + \frac{1}{4} = 2 \text{ Ans}$$

(ii) $\int_{2}^{3} \frac{x\,dx}{1+x^2}$

$$= \frac{1}{2}\int_{2}^{3} \frac{2x\,dx}{1+x^2}$$

$$= \frac{1}{2}\left[\text{In}(1+x^2)\right]_2^3 = \frac{1}{2}\left[\text{In}(1+3^2) - \text{In}(1+2^2)\right]$$

$$= \frac{1}{2}[\text{In}\,10 - \text{In}\,5] = \frac{1}{2}\text{In}\frac{10}{5}$$

$$= \frac{1}{2}\text{In}2 \text{ Ans}$$

$\text{In}\, x\, x - \int \frac{1}{x}(x)$

Example 3: Evaluated

(i) $\int_1^e \text{In}\, x\, dx$ (ii) $\int_{-3}^{3} \frac{dx}{3x^2+9}$ $x\,\text{In}\,x - x + c$

Solution:

(i) $\int_1^e \text{In}\, x\, dx$ Note: Integrate by parts.

$$= \left[x\,\text{In}\,x - x\right]_1^c = [\,(e\,\text{In}\,e - e) - (\text{In}1 - 1)]$$

$$= [(e - e) - (0 - 1)] \qquad \therefore \text{In}\, e = 1$$

$$= 1 \quad \text{Ans.}$$

(ii) $\int_{-3}^{3} \frac{dx}{x^2+9}$

$$= \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_{-3}^{3} = \frac{1}{3}\left[\tan^{-1}\frac{x}{3}\right]_{-3}^{3}$$

$$= \frac{1}{3}\left[\tan^{-1}\left(\frac{3}{3}\right) - \tan^{-1}\left(\frac{-3}{3}\right)\right]$$

$$= \frac{1}{3}\left[\tan^{-1}1 - \tan^{-1}(-1)\right]$$

$$= \frac{1}{3}\left[\frac{\pi}{4} - \left(\frac{-\pi}{4}\right)\right] = \frac{1}{3}\left[\frac{\pi}{4} + \frac{\pi}{4}\right] = \frac{1}{3}\left[\frac{\pi}{2}\right]$$

$$= \frac{\pi}{6} \quad \text{Ans}$$

Example 4: Calculate

(i) $\int_0^{\frac{\pi}{4}} \tan^4 x \sec^2 x\, dx$ (ii) $\int_0^{\frac{\pi}{3}} \frac{dx}{1 - \sin x}$

Solution:

(i) $$\int_0^{\frac{\pi}{4}} \tan^4 x \sec^2 x \, dx$$

$$= \left[\frac{\tan^5 x}{5}\right]_0^{\frac{\pi}{4}} = \frac{1}{5}\left[\tan^5\frac{\pi}{4} - \tan^5 0\right]$$

$$= \frac{1}{5}[(1)^5 - (0)^5] = \frac{1}{5}[1-0]$$

$$= \frac{1}{5} \quad \text{Ans.}$$

(ii) $$\int_0^{\frac{\pi}{3}} \frac{dx}{1-\sin x} = \int_0^{\frac{\pi}{3}} \frac{1+\sin x}{(1-\sin x)(1+\sin x)} dx$$

$$\int_0^{\frac{\pi}{3}} \frac{(1+\sin x)}{(1-\sin^2 x)} dx = \int_0^{\frac{\pi}{3}} \frac{1+\sin x}{\cos^2 x} dx$$

$$= \int_0^{\frac{\pi}{3}} \sec^2 x \, dx + \int_0^{\frac{\pi}{3}} \sec x.\tan x \, dx$$

$$= [\tan x]_0^{\frac{\pi}{3}} + [\sec x]_0^{\frac{\pi}{3}}$$

$$= \left[\tan\frac{\pi}{3} - \tan 0\right] + \left[\sec\frac{\pi}{3} - \sec 0\right]$$

$$= \sqrt{3} - 0 + 2 - 1 = \sqrt{3} + 1 \quad \text{Ans}$$

9.3 Integration by Substitution in Definite integral

The technique of change of variable may be used for evaluating definite integrals as well. However, great care must be exercised in evaluating the limits. There are two possible methods of procedure, and we illustrate each with an example.

Example 6: Evaluate

$$\int_1^2 (x+1)(x^2+2x+2)^{1/3}\,dx$$

Solution:

Let $u = x^2 + 2x + 2$, then $du = (2x + 2)\,dx$ substituting, we find the indefinite integral.

$$\int (x+1)(x^2+2x+2)^{1/3}\,dx = \int u^{1/3}\frac{1}{2}du = \frac{1}{2}\cdot\frac{3}{4}u^{4/3} + c = \frac{3}{8}u^{4/3} + c$$

Method-I: involves substituting back for u in terms of x, thereby obtaining for the definite Integral.

$$\int_1^2 (x+1)(x^2+2x+2)^{1/3}\,dx = \frac{3}{8}\Big[(x^2 + 2x + 2)^{\frac{4}{3}}\Big]_1^2$$

$$= \frac{3}{8}(10^{4/3} - 5^{4/3})$$

The next example will be done by the second method.

Example 7:

Evaluate $\displaystyle\int_{-2}^{0} x\sqrt{2x^2+1}\,dx$

Solution:

Let $u = 2x^2 + 1$, then $du = 4x\,dx$ Method-II involves changing the limits of define integral to value of u instead of values of x. Since $u = 2x^2 + 1$, we easily see that

When $x = -2$, $u = 9$

When $x = 0$, $u = 1$

Therefore we can write

$$\int_{-2}^{0} x\sqrt{2x^2+1}\,dx = \int_{-2}^{0} \sqrt{2x^2+1}\,(x\,dx)$$

$$= \int_9^1 U^{1/2}\frac{1}{4}du = \left[\frac{1}{4}\cdot\frac{2}{3}U^{3/2}\right]_9^1$$

$$= \frac{1}{6}(1^{3/2} - 9^{3/2}) = -\frac{13}{3}$$

Most of the time Method-II is simpler and shorter. Occasionally there are problems in which it pays to go back to the original variables before evaluating the limits of integration.

Exercise 9.1

Q.1: Calculate the following Integrals.

(i) $\int_1^3 \left(x - \frac{1}{x}\right) dx$ (ii) $\int_1^8 \frac{dx}{\sqrt[3]{x}}$ (iii) $\int_2^3 \frac{x}{1+x^2} dx$ S.B

(iv) $\int_0^3 \sqrt[3]{(3x-1)^2}\, dx$ (v) $\int_{-3}^{-1} \frac{dx}{(x-1)^2}$ (vi) $\int_0^2 \frac{x^2+1}{\sqrt{x^3+3x+1}} dx$

(vii) $\int_1^4 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$ (viii) $\int_0^a \frac{dx}{\sqrt{x+a}+\sqrt{x}}$

(ix) $\int_0^1 \frac{x}{\sqrt{1+x^2}} dx$ (x) $\int_1^3 \frac{dx}{x^2-16}$ (xi) $\int_0^{16} \frac{\sqrt{x}}{1+\sqrt{x}} dx$

Q.2: Calculate the following Integrals :

(i) $\int_0^{\pi/4} \frac{dx}{\cos^2 x}$ (ii) $\int_0^{\pi/4} \sin^2 x\, dx$ (iii) $\int_0^{\pi/4} \sin^3 x \cos x\, dx$

(iv) $\int_0^{\pi/3} \sin^3 x\, dx$ (v) $\int_0^{\pi/6} \frac{\tan x}{\cos^2 x} dx$ (vi) $\int_0^{\pi/2} \frac{dx}{1+\sin x}$

(vii) $\int_0^{\pi/3} \tan^3 x \sec x\, dx$ (viii) $\int_0^{\pi/4} (1 + \sec^2 x)\, dx$ (ix) $\int_0^{\pi/4} \tan^2 x\, dx$

Answers 9.1

Q.1: (i) $4 - \ln 3$ (ii) $\frac{9}{2}$ (iii) $\frac{1}{2}\ln 2$ (iv) $\frac{33}{5}$

(v) $\frac{1}{4}$ (vi) $\frac{2}{3}(\sqrt{15}-1)$ (vii) $\frac{20}{3}$ (viii) $\frac{4}{3}\sqrt{a}(\sqrt{2}-1)$

(ix) $\sqrt{2} - 1$ (x) $\frac{1}{8}\ln\frac{3}{35}$ (xi) $2(\ln 5 + 4)$

Q.2: (i) 1 (ii) $\frac{\pi-2}{8}$ (iii) $\frac{1}{16}$ (iv) $\frac{5}{24}$ (v) $\frac{1}{6}$

(vi) ∞ (vii) $\frac{4}{3}$ (viii) $\frac{\pi-4}{4}$ (ix) $\frac{4-\pi}{4}$

9.4 Applications to Area

Introduction:

The process of finding area bounded by the given curve is called quardrature

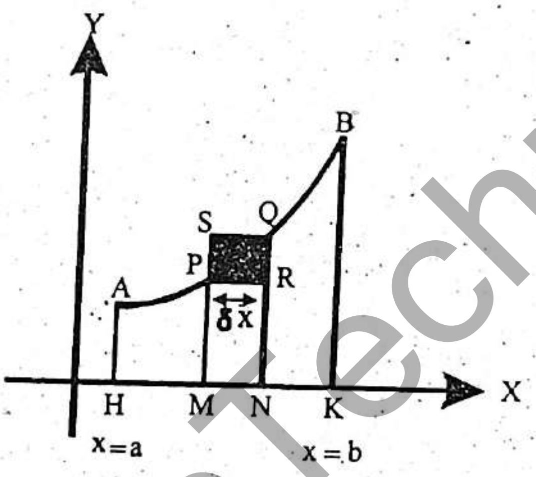
9.4.1 Area Bounded by the Curve

Let i. $f(x)$ be a continuous and single valued function of x in a interval [a,b].

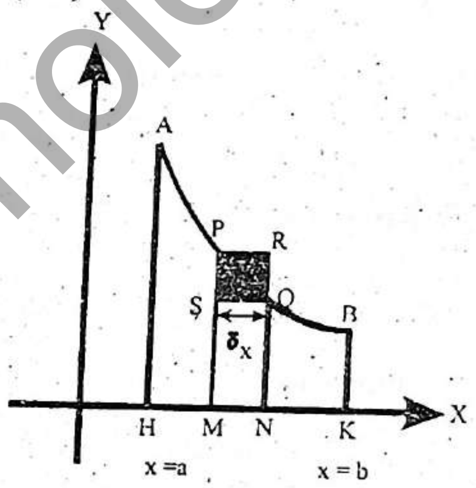
ii. a and b are finite and $a < b$.

Let AB be the curve $y = f(x)$, such that as a point move along it form A to B, either y increases constantly as in figure (7.1)

Or, y – decreases constantly as in figure (7.2)



(Fig 7.1)



(Fig 7.2)

Let AH and BK, be the ordinates, at x = a, x = b and p(x,y) be any point on the curve with ordinate PM. Then as x changes, the area AHMP also changes and is there the neighbourhood of P and let its ordinate be QN.

Then the area $\quad PMNQ = \delta A$

Complete the rectangle PRQS. In both the figure PMNQ lies between the areas of the rectangles PMNR and SMNQ.

i.e. δ A lies between the Area $y \cdot \delta x$ and $y + \delta y) \cdot \delta x$

and $\dfrac{\delta A}{\delta x}$ lies between y and $y + \delta y$.

In the limiting case as $\delta x \to 0$, $\delta y \to 0$, then

$$\frac{dA}{dx} = y \quad \Rightarrow \quad dA = y dx$$

Integrating to both the sides $A = \int_a^b y\,dx$

Similarly if the curve is in the form x = f(y) then $A = \int_a^b x\,dy$

The importance of this theorem results from the fact that it permits the evaluation of the limit of a sum of terms by integration and that this limit can be interpreted as the area under a curve. Specifically, the definite integral

$\int_a^b f(x)\,dx$ can be interpreted as the area bounded by the continuous positive curve y = f(x), the x – axis, and the lines x = a and x=b.

Solved examples:

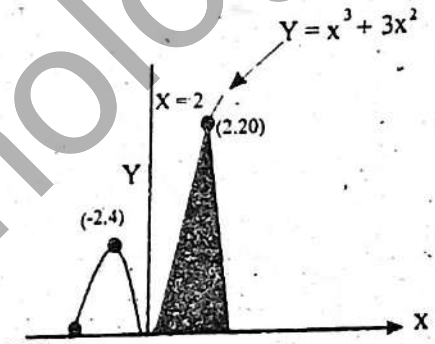


Fig.7.3

Examples1: Find the area bounded by the curve

$y = x^3 + 3x^2$

the x – axis, and the lines x = 0 and x = 2 **(see Fig 7.3)**

Solution : $A = \int_0^2 (x^3 + 3x^2)\,dx = \left[\frac{x^4}{4} + x^3\right]_0^2 = 12$

Examples 2: Find the area bounded by the curve

$x^2y = x^2 - 4$

the x – axis, and the lines x = 2 and x = 4 **(see Fig. 7.4)**

Solution :

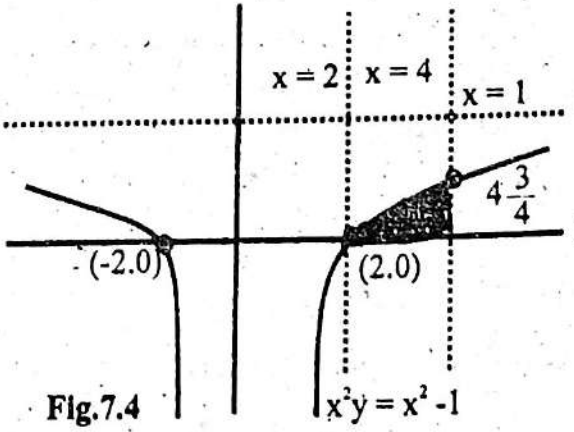


Fig.7.4

$$A = \int_2^4 \frac{x^2-4}{x^2}\,dx = \int_2^4 \left(1-\frac{4}{x^2}\right) dx$$

$$= \left[x + 4x^{-1}\right]_2^4$$

$$= 4 + 1 - 2 - 2$$

$$= 1$$

Examples 3: Find the area in the first quadrant bounded by x – axis and the curve. $y = 6x + x^2 - x^3$ (see Fig. 7.5)

Solution : For limiting values put $6x + x^2 - x^3 = 0$

$x = 0$ or $6 + x - x^2 = 0 \Rightarrow x = -2, 0, 3$

$$A = \int_0^3 (6x + x^2 - x^3)\,dx = \left[3x^2 + \frac{x^3}{3} - \frac{x^4}{4}\right]_0^3 = 27 + 9 - \frac{81}{4} = 15\frac{3}{4}$$

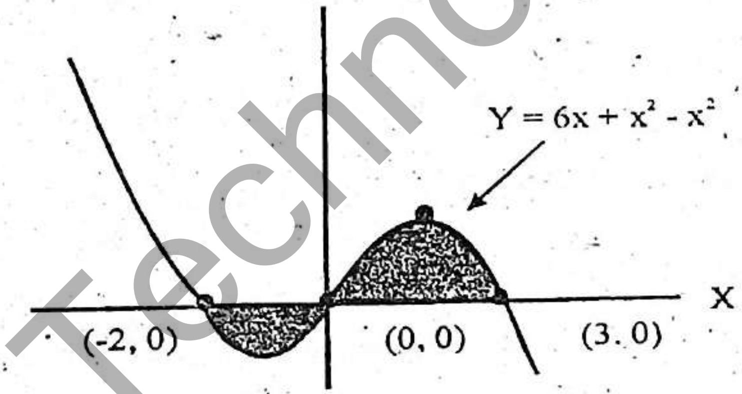


Fig.7. 5

Examples 4: Find the total area between the porabola.

$y = x^2 - 4x$

the x – axis, and the line x = –2 (see Fig. 7.6) d

Solution : $A = \int_{-2}^{0} (x^2 - 4x)dx = \left| \frac{x^3}{3} - 2x^2 \right|_{-2}^{0}$

$$= -\left(-\frac{8}{3} - 8\right) = 10\frac{2}{3}$$

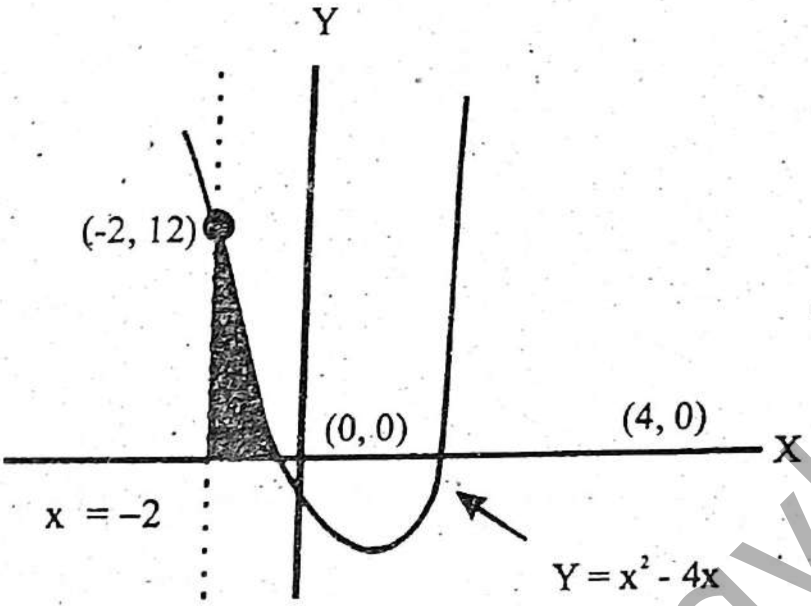


Fig.7. 6

Examples 5: Find area of the region bounded by the curve $y = x^2$, x-axis from $x = -3$ to $x = 1$

Solution : $A = \int_{-3}^{1} y\,dx = \int_{-3}^{1} x^2\,dx = (1/3)\,|x^3|^{1}_{-3}$

$$= \frac{1}{3}\left(1^3 - (-3)^3\right) = \frac{1}{3}(28) = \frac{28}{3} \text{ Sq.unit}$$

9.4.2 Interpretation of Negative Areas

In the definition of area given above.

$$A = \int_a^b f(x)\,dx$$

$f(x)$ is assumed to be a continuous positive function between a and b. if $f(x)$ is negative that is, if the curve $y = f(x)$ lies below the x – axis, between a and b, then the value of the integral.

$$A = \int_a^b f(x)\,dx \quad \text{is negative.}$$

Such areas below the x – axis are called negative areas, the total absolute area between a curve, the x – axis, and two ordinates is given by

$$\text{Total area} = \Sigma\,(\text{positive areas}) - \Sigma\,(\text{negative areas}) = \int_a^c f(x)\,dx - \int_c^b f(x)dx$$

Note: This is equivalent to say that area is equal to the absolute value of the integral and thus is always positive.

Examples 6: Find the area bounded by the curve.

$$y = 2x + x^2 - x^3$$

the x – axis, and the lines x = – 1 and x = 1 (see **Fig. 7. 7**)

Solution :

$$A = \int_0^1 (2x + x^2 - x^3)\,dx - \int_{-1}^0 (2x + x^2 - x^3)\,dx$$

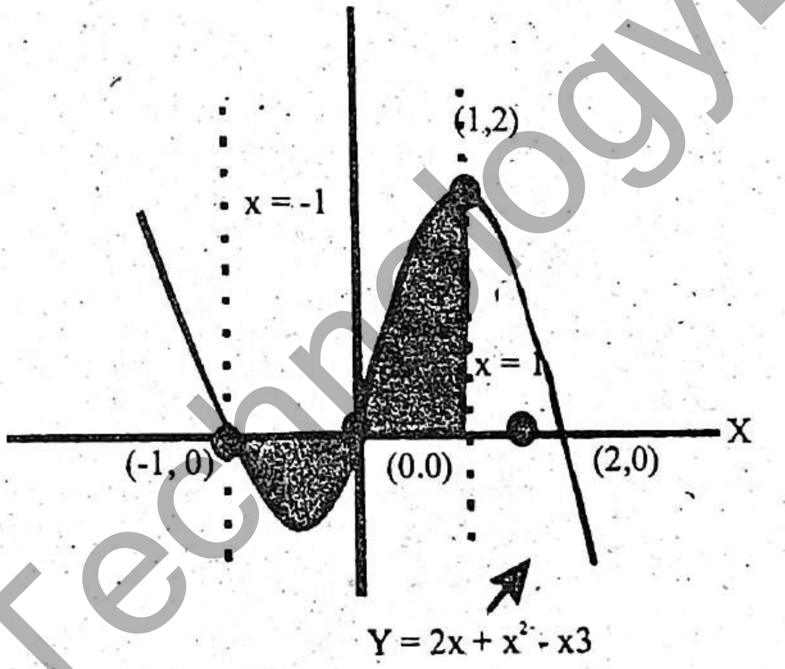


Fig.7.7

$$= \left[x^2 + \frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 - \left[x^2 + \frac{x^3}{3} - \frac{x^4}{4}\right]_{-1}^0$$

$$= \left[\left(1 + \frac{1}{3} - \frac{1}{4}\right) - 0\right] - \left[0 - \left(1 - \frac{1}{3} - \frac{1}{4}\right)\right]$$

$$= \frac{13}{12} - \left(-\frac{5}{12}\right) \text{ (positive area minus negative area)}$$

$$= \frac{3}{2}$$

Examples 7:

Find area bounded by the curve:

$$y = x^3 - 4x$$

and x – axis (**see Fig. 7.8**)

Solution :

$$A = \int_{-2}^{0} (x^3 - 4x)\,dx - \int_{0}^{2} (x^3 - 4x)\,dx$$

$$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} - \left[\frac{x^4}{4} - 2x^2\right]_{0}^{2}$$

$$= [0 - (4 - 8)] - [(4 - 8) - 0]$$

$$= 4 - (-4) = 8$$

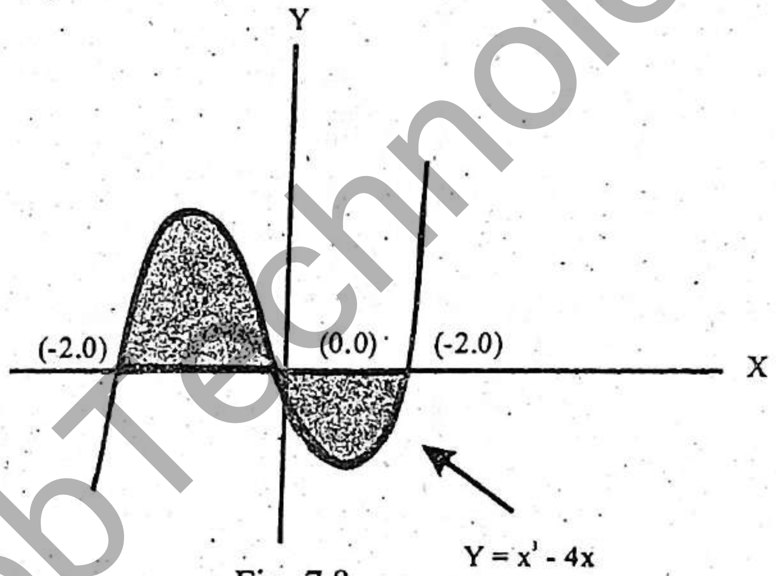


Fig. 7.8

9.4.3 Area Between Two curves

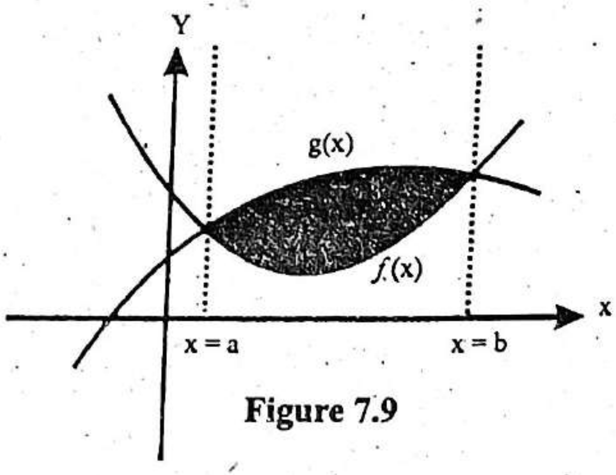


Figure 7.9

Suppose that the area to be evaluated is between the curves $y_1 = f(x)$ and $y_2 = g(x)$ and the lines $x = a$ & $x = b$ and that(for definiteness) $f(x) \leq g(x)$ for $a \leq x \leq b$ (**see Fig. 7.9**). Then

$$A = \int_{a}^{b} [g(x) - f(x)]\,dx$$

Note that this formula includes negative area (with appropriate signs) in the total area between the curves.

Example 8:

Find the area bounded by the curves

$y = x^2$ and $y = x$ **(See Fig. 7.10)**

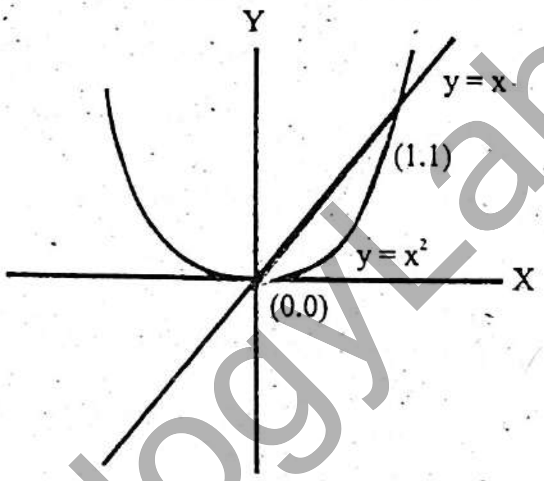


Fig. 7.10

Solution : Find the points of intersection of the curve

$$y = x^2 = x$$

$$x(x-1) = 0$$

$$x = 0, 1$$

If $x = 0$, $y = 0$, if $x = 1$, $y = 1$

Thus

$$A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}$$

Examples 9:

Find the area bounded by the curves

$y = x^3$ and $y = 4x^2$

(see Fig.7.11)

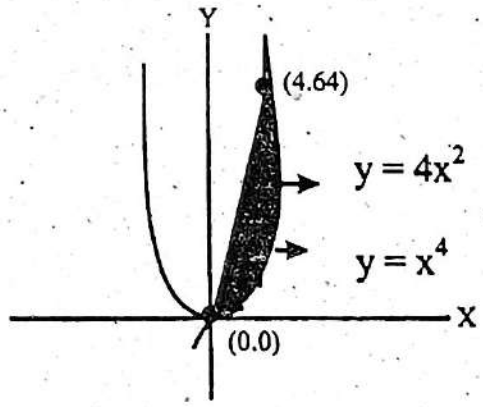


Fig. 7.11

Solution : Finding the points of intersection of the curves

$$y = x^3 = 4x^2$$

$$x^2(x-4) = 0 \quad \Rightarrow \quad x = 0, 4$$

If $x = 0$, $y = 0$, if $x = 4$, $y = 64$ Thus

$$A = \int_0^4 (4x^2 - x^3)\,dx = \left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_0^4$$

$$= \frac{256}{3} - 64 \quad = \quad 21\frac{1}{3}$$

Note: In evaluating an area it is theoretically of no importance whether horizontal or vertical elements of area are used, the choice depends on the number and difficulty of the integrals needed to determine the particular area. In the above example either choice involves approximately the same amount of calculation.

Examples 10:

The area bounded by the curves

$$y = x^2 \qquad \text{and} \qquad y = x$$

Solution :

Could have been determined using horizontal elements of area as follows:

since, $x = \sqrt{y}$ and $x = y$ for the point intersection of the curves

$$y = \sqrt{y} \Rightarrow y^2 = y \Rightarrow y(y-1) = 0 \Rightarrow y = 0, 1$$

$$A = \int_0^1 (y^{1/2} - y)\, dy = \left[\frac{2y^{3/2}}{3} - \frac{y^2}{2}\right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

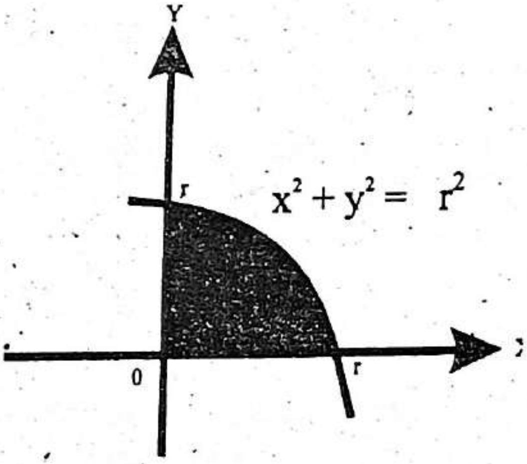
Examples 11:

Show that area of a circle of radius r is πr^2.

Solution: The equation of a circle of radius r centered at the origin is

$$x^2 + y^2 = r^2$$

$$y = \pm\sqrt{r^2 - x^2}$$



Fg.7.12

A plot of one-quarter of the circle is shown in Fig.7.12. It is necessary to work with a quarter circle only.

The equation for this quarter circle is

$$y = \sqrt{r^2 - x^2} \qquad \text{for } 0 \le x \le r$$

$$A = \int_0^r y\, dx = \int_0^r \sqrt{r^2 - x^2}\, dx \qquad \text{(area in the first quadrant)}$$

Put $\quad x = r\sin\theta, \qquad dx = r\cos\theta\, d\theta$

When $x = 0\ ,\quad \theta = 0$

When $x = r\ ,\quad \theta = \frac{\pi}{2}$

$$A = \int_0^{\pi/2,} \sqrt{r^2 - r^2\sin^2\theta}\ .\ r\cos\theta\ d\theta \ = \int_0^{\pi/2} r^2\cos^2\theta\ d\theta$$

$$= \frac{r^2}{2}\int_0^{\pi/2} (1+\cos 2\theta)\ d\theta$$

$$= \frac{r^2}{2}\left|\theta + \frac{1}{2}\sin 2\theta\right|_0^{\pi/2}$$

$$= \frac{r^2}{2}\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right) - \left(0 + \frac{1}{2}\text{Sin } 0\right)\right]$$

$$A = \frac{\pi r^2}{4}$$

Area of the circle $= 4A = 4\left(\frac{\pi r^2}{4}\right) = \pi r^2$ square unit

Exercise 9.2

Q.1: Find the total area of the region by the curv $y = x^3 - 4x$ and x– axis

Q.2: Find area bounded by the line $3x - y - 3 = 0$ and $x = 1$ & $x = 5$.

Q.3: Find area between the curve $y = 3x^2 - 3$ and x – axis.

Q.4: Prove that the area bounded by the curve $3ay^2 = x(x - a)^2$ is $\frac{8a^2}{15\sqrt{3}}$.

Q.5: Find area bounded by $y = 3x$, $y = x^2$ between $x = 1$ and $x = 3$.

Q.6: Compute area bounded by the curve $y = \sqrt{x}$ and $y = x^2$.

Q.7: Find area of the region enclosed by curve $y = 3 - x^2$ and line $y = -x + 1$.

Q.8: Find area of the region enclosed by parabola $y = 2 - x^2$ and line $y = -x$.

Q.9: Find area between the curves $y = x^2$ and $y = -x^2 + 4x$.

Q.10: Compute area of the region bounded by curve $y = x^4$ and line $y = 8x$.

Q.11: Find area bounded by $y^2 - 3x + 3 = 0$ and the line $x = 4$

Q.12: Find area between the curve $y = x^2 - 4x$, the x-axis and the line $x = -2$.

Answers 9.2

Q.1: 8 sq.units

Q.2: 24 sq.units

Q.3: 4 sq.units

Q.5: $\frac{10}{3}$ sq.units

Q.6: $\frac{1}{3}$ sq.units

Q.7: $\frac{9}{2}$ sq.unit

Q.8: $\frac{9}{2}$ sq.units

Q.9: $\frac{8}{3}$ sq.units

Q.10: $\frac{48}{5}$ sq.units

Q.11: 12 sq.unit

Q.12: $\frac{64}{3}$ sq.units

Short Questions

Write the short answers of the following:

Q.1: Evaluate $\int_{0}^{3} \sqrt[3]{(3x-1)^2}\, dx$

Q.2: Evaluate $\int_{1}^{3} \frac{1}{x+1}\, dx$

Q3: Evaluate $\int_{\pi/6}^{\pi/3} \sin 2x \quad dx$

Q.4: Evaluate $\int_{\pi/6}^{\pi/3} \text{cosec}^2 x \, dx$

Q.5: Evaluate $\int_{0}^{\pi/4} \sec x \tan x \, dx$

Q.6: Evaluate $\int_{0}^{b} x^3 \text{Cos}\, x^4 \, dx$

Q.7: Evaluate $\int_{0}^{\pi/2} \frac{\cos x}{3 + 4\sin x}\, dx$

Q.8: Evaluate $\int_{0}^{\pi/6} 2\sin 2x \, dx$

S.P

Q.9: Evaluate $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx$

Q.10: Evaluate $\int_{0}^{\pi/6} \sec^2 x \, dx$

Q.11: Evaluate $\int_{0}^{1} \frac{1}{\sqrt{1-x^2}}\, d$

P

Q.12: Evaluate $\int_{1}^{\sqrt{3}} \frac{1}{1+x^2}\,dx$

Q.13: Evaluate $\int_{1}^{3} \frac{2x-1}{x^2-x+1}\,dx$

Q.14: Evaluate $\int_{0}^{\pi/6} \text{Sin } x \cos x\,dx$

Q.15: Find value of $\int_{0}^{\pi/4} \cos^2 x\,dx$

Q.16: Find value of $\int_{0}^{\pi/2} \tan^2 x\,dx$

Q.17: Find value of $\int_{0}^{\pi/2} \sin^2 x \cos x\,dx$

Q.18: Find value of $\int_{0}^{\pi} x \cos x\,dx$

Q.19: Find value of $\int_{0}^{1} x e^x\,dx$

Answers

Q.1 10	Q.2 ln 2	Q.3 $\frac{1}{2}$	Q.4 $\frac{2}{\sqrt{3}}$
Q.5 $\sqrt{2}-1$	Q.6 $\frac{1}{4}\sin b^4$	Q.7 $\frac{1}{4}\ell n\frac{7}{3}$	Q.8 $\frac{1}{2}$
Q.9 2	Q.10 $\frac{1}{\sqrt{3}}$	Q.11 $\frac{\pi}{12}$	Q.12 $\frac{\pi}{3}$
Q.13 $\ell n7$	Q.14 $\frac{1}{8}$	Q.15 $\frac{1}{8}(\pi+2)$	Q.16 ∞
Q.17 $\frac{1}{3}$	Q.18 -2	Q.19 1	

Objective Type Questions

Q.1: Encircle the correct one, of the given answers in each item.

1 $\int_0^1 dx =$

(a) −1 (b) 0 [✓ (c) 1] (d) 2

2. $\int_0^{\pi/4} \sec^2 x\,dx =$

[✓ (a) 1] (b) 2 (c) Zero (d) 3

3. If lower and upper limits of $\int_a^b f(x)\,dx$ are replaced by each other then we get

(a) $\int_a^b f(x)\,dx$ [✓ (b) $-\int_b^a f(x)\,dx$]

(c) $\pm\int_a^b f(x)\,dx$ (d) $\int_b^a f(x)\,dx$

4. $\int_2^3 \frac{2x\,dx}{x^2+1}$

[✓ (a) $\ell n\,2$] (b) $\ell n\,10 - \ell n\,6$

(c) $\ell n\,(x^2+1)$ (d) $\ell n\,10 + \ell n\,6$

5. $\int_1^3 e^{2x}\,dx =$

(a) $e^6 - e^2$ [✓ (b) $\frac{1}{2}(e^6 - e^2)$]

(c) $\frac{e^{2x}}{2}$ (d) $\frac{1}{2}(e^6 + e^2)$

6. $\int_1^2 3x^2\,dx =$

[✓ (a) 7] (b) 8 (c) 6 (d) 9

7. $\int_0^1 \frac{1}{x^2+1}\,dx =$

[✓ (a) $\frac{\pi}{4}$] (b) 0 (c) 1 (d) $-\frac{\pi}{4}$

S.P. ✓ 8. $\int_0^1 \frac{1}{x\sqrt{x^2-1}}\,dx =$

(a) -1 (b) 1 (c) 0 [✓ (d) $-\frac{\pi}{2}$]

9. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\text{cosec}^2 x}{\cot x}\,dx =$

(a) 2 (b) $\sqrt{3}$ (c) $\frac{1}{\sqrt{3}}$ [✓ (d) $\ln\frac{1}{\sqrt{3}}$]

S.P. ✓ 10. $\int_0^{\frac{\pi}{2}} \cot x\,dx =$

(a) -1 (b) 1 [✓ (c) 0] (d) $\frac{\pi}{2}$

Answers

Q.1:

i	c	ii	a	iii	b	iv	a	v	b
vi	a	vii	a	viii	d	ix	d	x	c

Chapter 10

Differential Equations

10.1 Introduction:

An equation involving derivatives or differentials is called a **differenti equation.**

For example:

$$\frac{dy}{dx}+2x=0$$

$$\frac{d^2y}{dx^2}-\frac{2dy}{x\,dx}+\frac{2y}{x^2}=0$$

$$dy+3x^2(1+y^2)\,dx=0$$

Order of Differential Equation:

The "Order" of a differential equation is the order of the highest derivati which appears in the differential equation.

For example: $\frac{dy}{dx}+2x=0$ is a differential equation with order 1.

$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-4)y=0$ is a differential equation with order 2.

Degree of Differential Equation:

The "Degree" of a differential equation is the highest power of the highe order derivative in the equation, after making it free from radicals and fractions.

For example: $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$ has degree one

$\frac{d^2y}{dx^2}+\left(\frac{d^2y}{dx^2}\right)^2+y=0$ has degree two

$\frac{dy}{dx}+\frac{x^2}{dy/dx}=x$ has degree two because , $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + x^2 = 0$

Solution of Differential Equations :

A solution of Differential equation is a relation between dependent and independent variables, which is free of derivatives. If this solution is substituted in the given equation, it is reduced to an identity.

There are many types of equations, the solution of which are readily found. For example, an equation such as

$\frac{dy}{dx} = f(x)$ is a differential equation and the integral

$y = \int f(x)\,dx + c$ is its solution, where c is arbitrary constant.

More generally equation

$\frac{d^2y}{dx^2} = g(x)$ gives its solution, when integration 2 times.

In fact integration is an example of a process for solving differential equations.

For example $y = ax^2 + bx$ is a solution of

$$\frac{d^2y}{dx^2} - \frac{2dy}{x\,dx} + \frac{2y}{x^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

because on substitution of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1) we get

$$\frac{d^2y}{dx^2} - \frac{2dy}{x\,dx} + \frac{2y}{x^2} = 2a - \frac{2}{x}(2ax + b) + \frac{2}{x^2}(ax^2 + bx)$$

$$= 2a - 4a - \frac{2b}{x} + 2a + \frac{2b}{x} = 0$$

General Solution:

A solution which contains the same number of arbitrary constants as the order of the differential equation is called General Solution.

For example a general solution of

$$\frac{d^2y}{dx^2} + c^2y = 0$$

is $y = P\sin cx + Q\cos cx$, where P and Q are arbitrary constants

Particular Solution:

A solution obtained from the general solution by giving particular numerical values to arbitrary constants, by applying the initial and boundary conditions, is called particular solution.

10.2 Differential Equation of First Order and First Degree:

A differential equation of first order and first degree may be written in the differential form $M(x,y)\,dx + N(x,y)\,dy = 0$

10.3 First Order Differential Equation:

There are many ways to solve first order differential equations. Some important types of the equations and method to solve them are

1. Equations with variables separable.
2. Homogenous first order equations.
3. Linear first order equations.
4. Exact first order equations

Note : We only use Variables Separable method to solve the Differential Equations. The other three methods are out of scope of this book.

10.3.1 Equations with variables separable:

A first order differential equation, which by algebraic manipulations, can be reduced to the form.

$$f(x)\,dx = g(y)\,dy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

is called variables separable form.

It is because the variables x and y can be separated from each other in such a way that x appears only in the coefficient of dx and y appears only in the coefficient of dy. Such an equation is solved by integrating (1)

$$\int f(x)\,dx = \int g(y)\,dy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Some other forms which should be recognized as being separable are given below:

$$f(x)\,G(y)\,dx = g(y)\,F(x)\,dy \ldots\ldots\ldots\ldots\ldots(3)$$

$$\frac{dy}{dx} = M(x)\,N(y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Solution of (3) can be found as

$$\int \frac{f(x)}{F(x)}\,dx = \int \frac{g(y)}{G(y)}\,dy \ldots\ldots\ldots\ldots\ldots(5)$$

And Solution of (4) can be found as

$$\int \frac{dy}{N(y)} = \int M(x)\,dx \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Now we solve some examples:

Example 1:

Solve the equations (i) $dy = \sec^2 x\ dx$ (i) $y\,e^{x+y}\,dy = dx$

Solution:

(i) $dy = \sec^2 x\ dx$

$$\int dy = \int \sec^2 x\ dx$$

$$y = \tan x + c$$

(ii) $y.\, e^x .\, e^y\, dy = dx$

Separating variables we get

$$y\, e^y\, dy = e^{-x}\, dx$$

$$\int y\, e^y dy = \int e^{-x} dx$$

$$y \int e^y dy - \int \left(\frac{d}{dy} y. \int e^y dy \right) dy = \int e^{-x} dx$$

$$y e^y - \int e^y\, dy = \frac{e^{-x}}{-1} + c$$

$$y e^y - e^y = \frac{-1}{e^x} + C$$

$$(y-1)\, c^y .\, e^x = -1 + C$$

$$e^{x+y}(y-1) + 1 = C$$

L.H.S side is being integrated by parts

Example 2:

Find the solution of (i) $\frac{dy}{dx} = -4xy^2$ (ii) $\frac{dy}{dx} = \frac{x}{y^2}$ (iii) $\frac{dy}{dx} = \frac{y}{4 + x^2}$

Solution:

Separating variables we get

(i) $\frac{dy}{y^2} = -4x\, dx$

$\int y^{-2} d\, y = -4 \int x dx$

$\frac{y^{-1}}{-1} = -4\frac{x^2}{2}$

$-\frac{1}{y} = -2x^2 + c$

$y = \frac{1}{2x^2 - c}$

(ii) $\frac{dy}{dx} = \frac{x}{y^2}$

Solution:

$$y^2 dy = x\, dx$$
$$\int y^2 dy = \int x\, dx$$
$$\frac{1}{3} y^3 = \frac{1}{2} x^2 + c$$
$$y^3 = \frac{3}{2} x^2 + c$$

(iii) $\frac{dy}{dx} = \frac{y}{4 + x^2}$

Solution:

$$\frac{dy}{y} = \frac{1}{4 + x^2}\, dx$$

$$\int \frac{dy}{y} = \int \frac{1}{4 + x^2}\, dx$$

$$\ln y = \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + c$$

Example 3:

Find the particular solution of

$dy = x(2y\, dx - x\, dy)$ subject to the conditions $x = 1, y = 4$

Solution $dy = 2xy\,dx - x^2\,dy$

$$dy + x^2\,dy = 2xy\,dx$$

$$(1 + x^2)\,dy = 2xy\,dx$$

$$\int \frac{dy}{y} = \int \frac{2x}{1 + x^2}\,dx$$

$$\text{In}\,y = \text{In}\,(1 + x^2) + \text{In}\,c$$

Where $\text{In}\,c$ is constant of integration

$$\text{In}\,y = \text{In}\left[(1 + x^2)\,c\right]$$

$$y = (1 + x^2)\,c \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now put $x = 1$ and $y = 4$

$$4 = (1 + (1)^2)\,c$$

$$4 = (1 + 1)\,c = 2c$$

$$c = 2$$

Equation (1) becomes $y = 2(1 + x^2)$, which is the required solution.

Example 4 :

A particle is moving in a straight line and its acceleration is given by

$$a = 2t - 9$$

(i) Find v (velocity) in terms of t if $v = 15$ m/sec, when $t = 0$

(ii) Find s (distance) in terms of t if $s = 0$, when $t = 0$

Solution : Given that $a = 2t - 9$

Then, $\dfrac{dv}{dt} = 2t - 9$ $\quad\left[\text{because } a = \dfrac{dv}{dt}\right]$

$$dv = (2t - 9)\,dt$$

integrating both sides, we get

$$\int dv = \int (2t - 9)\,dt$$

$$v = t^2 - 9t + c_1 \text{------------------------------}(1)$$

Apply the first initial value condition

$$15 = 0 - 0 + c_1 \quad \Rightarrow \quad c_1 = 15$$

The equation (1) becomes

$v = t^2 - 9t + 15$ which is the solution of (1)

Now, $\frac{ds}{dt} = t^2 - 9t + 15$ [because $v = \frac{ds}{dt}$]

$ds = (t^2 - 9t + 15)\,dt$ --(2)

integrating equation (2) to both sides

$$\int ds = \int (t^2 - 9t + 15)\,dt$$

$$s = \frac{t^3}{3} - 9\frac{t^2}{2} + 15t + c_2$$

Apply the second initial value condition , we get

$0 = 0 - 0 + 0 + c_2 \Rightarrow c_2 = 0$

Thus , $s = \frac{t^3}{3} - 9\frac{t^2}{2} + 15t$

which is the solution of (2)

Exercise 10

Find the general solution of each of the following:

Q.1: $xdy = 3y\,dx$

Q.2: $y\,dy = 2(xy + x)\,dx$

Q.3: $x\frac{dy}{dx} = y^2 - 3y + 2$

Q.4: $y\,dx - xdy = x\,(dy - y\,dx)$

Q.5: $3x^2(1 + y^2)\,dx = dy$

Q.6: $y\,dx = 2(xy + x)\,dy$

Q.7: $dx + ydy = x^2y\,dy$

Q.8: $dx + xy\,dy = y^2dx + y\,dy$

Q.9: $\sin x \sec x\,dx = \sin y \sec y\,dy$

Q.10: $(x^2 + 1)\,dx = xy\,(y + 1)\,dy$

Q.11: $(x + 1)\frac{dy}{dx} = x\,(y^2 + 1)$

Q.12: $x^2 y\frac{dy}{dx} = (1 + x)\operatorname{cosec} y$

Q.13: $x^2\frac{dy}{dx} = \frac{1}{y^2 + \sqrt{y}}$

Q.14; $(y + xy)\,dx + (x - xy^2)\,dy = 0$

Q15: $(e^x + e^{-x})\frac{dy}{dx} = (e^x - e^{-x})$

Find the particular solution satisfying the given boundary conditions

Q.16: $2x\,dx - dy = x\,(xdy - y\,dx)$, $x = -3, y = 1$

Q.17: $3x^2y^2\,dx + y^2\,dx + dy = 0$, $x = 2, y = 1$

Q.18 : A particle is moving in a straight line and its acceleration is given by

$$a = 4t + 9$$

(i) Find v (velocity) in terms of t if v = 15 m/sec , when t = 0

(ii) Find s (distance) in terms of t if s = 0 , when t = 0

Answers

Q.1: $y = cx^3$

Q.2: $y - \ln(y+1) = x^2 + C$

Q.3: $y = xc(y-1) + 2$

Q.4: $y^2 = xe^{x-c}$

Q.5: $y = \tan(x^3 - C)$

Q.6: $y^2 = x\,e^{-2y-c}$

Q.7: $y^2 = \ln\dfrac{x-1}{x+1} + C$

Q.8: $y^2 = C^2(x-1)^2 + 1$

Q.9: $\sec x = C \sec y$

Q.10: $3x^2 + 6\ln x = 2y^3 + 3y^2 + C$

Q.11: $\tan^{-1}y = x - \ln(x+1) + C$

Q.12: $-y\cos y + \sin y = -\dfrac{1}{x} + \ln x + C$

Q.13: $\dfrac{1}{3}y^3 + \dfrac{2}{3}y^{3/2} = -\dfrac{1}{x} + C$

Q14: $y^2 = 2\ln(xy) + 2x + c$

Q15: $y = \ln(e^x + e^{-x}) + c$

Q.16: $x^2 + 1 = \dfrac{10}{9}(2+y)^2$

Q.17: $y = \dfrac{1}{x^3 + x - 9}$

Q.18: $v = 2t^2 + 3t + 15$, $s = \dfrac{2}{3}t^3 + \dfrac{3}{2}t^2 + 15t$

Short Questions

Write the short answers of the following:

Q.1: Define the Differential equation with example

Q.2: Define the Order of differential equation with example

Q.3: Define the Degree of differential equation with example

Q.4: What is general solution of differential equation?

Q.5: What is the particular solution of differential equation?

Q.6: Write down the order and degree of the following differential equations:

(a) $x\,dy + y\,dx = 0$ (b) $\frac{dy}{dx} + 2y = 0$

(c) $L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{c} = \omega E \sin \omega t$ (d) $\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$

Q.7: Find the order and degree of the following differential equations:

(a) $\frac{dy}{dx} - \frac{y}{dy/dx} = x^2$ (b) $\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^2 = 0$

(c) $\left(\frac{d^2y}{dx^2}\right)^3 - \left(\frac{d^3y}{dx^3}\right)^2 = y$ (d) $\sqrt[4]{1 + \left(\frac{d^2y}{dx^2}\right)^3} = \frac{d^3y}{dx^3}$

Q.8: Show that, $y = ce^{-2x}$ is the solution of differential equation $\frac{dy}{dx} + 2y = 0$.

Q.9: Show that $y = c\,e^{x^2}$ is the solution of differential equation $\frac{1}{x}\frac{dy}{dx} - 2y = 0$

Q.10: Solve the differential equation $\frac{dy}{dx} = \frac{2x+3}{4y-5}$

Q.11: Solve the differential equation $x^2\frac{dy}{dx} = \cos^2 y$

Q.12: Find the solution of $\frac{dy}{dx} = -\sin x + 3x^2$

Q.13: Find the solution of $\frac{dy}{dx} = 1 + x + y + xy$

Q.14: Find the solution of $\cos^2 x\frac{dy}{dx} + \cos^2 y = 0$

Q.15: Find the solution of $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

Q.16: Solve the equation $(1-x)\,dy = (1+y)\,dx$

Q.17: Find the solution of $dy = e^{x+y}\,dx$

Q.18: Find the solution of $(e^x - e^{-x})dy = (e^x + e^{-x})\,dx$.

Q.19: Find the particular solution of the equ. $\frac{dy}{dx} = xy$, given $y = 1$, when $x = 0$.

Q.20: Find the particular solution of the equation $2x\,dy + dx = dy$ given $y = 0$ when $x = 3$.

Q.21: The particular solution of $\frac{dy}{dx} = 2xy$, given $y = 1$, when $x = 0$.

Answers

Q6.(a) $O = 1, D = 1$ (b) $O = 1, D = 1$ (c) $O = 2, D = 1$ (d) $O = 2, D = 2$

Q7.(a) $O = 1, D = 2$ (b) $O = 3, D = 1$ (c) $O = 3, D = 2$ (d) $O = 3, D = 4$

Q10. $x^2 - 2y^2 + 3x + 5y + c = 0$ **Q11.** $\tan y + x^{-1} = c$

Q12. $y = \cos x + x^3 + c$ **Q13.** $\ln(1 + y) = x + \frac{1}{2}x^2 + c$ **Q14.** $\tan y + \tan x = c$

Q15. $\text{Sin}^{-1}y = \sin^{-1}x + c$ **Q16.** $(1 + y)(1 - x) = c$ **Q17.** $e^x + e^{-y} = c$

Q18. $y = \ln(e^x - e^{-x}) + c$ **Q19.** $y = e^{1/2x^2}$ **Q20.** $y + \frac{1}{2}\ln(2x - 1) = \frac{1}{2}\ln 5$

Q21. $y = e^{x^2}$

Objective Types Questions

Q.1: Encircle the correct one of the given answers in each item.

1. An equation involving one or more derivatives of a function is called.

   (a) Quadratic (b) Linear [✓ (c) Differential] (d) Cubic

2. Order of differential equation $\left(\frac{d^3y}{dx^3}\right)^2 + \frac{dy}{dx} + y = 0$ is

   [✓ (a) 2] (b) 1 (c) 0 (d) 3

S.Pa

3. Degree of differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$ is

   (a) 3 (b) 2 (c) 0 [✓ (d) 1]

4. Solution of differential equation $\frac{dy}{dx} = 1$ is

   (a) $y = x$ [✓ (b) $y = x + c$] (c) $y = c$ (d) $y = x^2 + c$

S.P

5. Solution of differential equation $xdy + ydx = 0$ is

   (a) $y = cx$ (b) $y = x$ [✓ (c) $xy = c$] (d) $y^2 = x^2 + c$

6. Solution of differential equation $\frac{dy}{dx} = -y$ is

   [✓ (a) $y = Ce^{-x}$] (b) $y = ce^{x}$ (c) $y = e^{x} + c$ (d) $y = e^{-x} + c$

7. If $x\,dy + y\,dx = 0$ is the differential equation, Then its variables separable form is

   (a) $y\,dy + x\,dx = 0$ (b) $\frac{1}{y}dy = \frac{1}{x}dx$ [✓ (c) $\frac{1}{y}dy = -\frac{1}{x}$] (d) $x\,dy = -y\,dx$

8. $y = \ln x + c$ is the solution of differential equation

   [✓ (a) $x\,dy = dx$] (b) $x\,dx = dy$ (c) $dy = \frac{1}{x}$ (d) $dy = dx$

9. Order of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = o$ is

   (a) 1 [✓ (b) 2] (c) 0 (d) 3

10. Degree of differential equation $x\left(\frac{d^3y}{dx^3}\right)^2 = 1$ is

    (a) 0 (b) 1 [✓ (c) 2] (d) 3

ANSWERS

Q.1:

1	c	2.	a	3.	d	4.	b	5	c
6.	a	7	c	8	a	9	b	10.	c

Chapter 11

Fourier Series

11.1 Introduction:

It was J.B. Fourier who used such expansions in different types of problems. A specific infinite trigonometric series used by him was of the form.

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n x + b_n \sin nx)$$

As trigonometric functions used in this series are periodic, therefore this series is a periodic function of period 2π.

This series is of fundamental importance in the study of physical systems subjected to periodic disturbances. It has waste applications in the present scientific age. For example, the voltage impressed on an electrical circuit might consist of a series of pulses or the disturbing influence acting on mechanical system might be a force of constant magnitude whose direction is periodically and instantaneously reversed. This theory is also applicable in many types of engineering problems. Its most important application is in the analysis of the behavior of physical systems subjected to periodic disturbances.

Before going to Fourier Series, some important terms are being explained.

11.2 Periodic Function:

If we add or substract a constant from the argument of a function, it remains with no change or we say $f(x) = f(x \pm A) \ \forall \ x$

Here A is called period of the function. Period of sin x and cos x is same, which is 2π.

11.3 Even Functions:

If for a function $f(x)$

$f(-x) = f(x)$

Then f(x) is called an even function.

For example, $f(x) = x^2$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

And for $f(x) = \cos x$

$$f(-x) = \cos(-x) = \cos x = f(x)$$

Therefore both x^2 and cos x are even functions of x

11.4 Odd Function:

If for a function $f(x)$

$$f(-x) = -f(x)$$

Then f (x) is called an odd function, for example

$$f(x) = x^3$$

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

And for $f(x) = \sin x$

$$f(-x) = \sin(-x) = -\sin x = -f(x)$$

Therefore both x^3 and sin x are odd functions of x

11.5 Fourier Function:

A trigonometric series of the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

Is called Fourier series

Suppose $f(x)$ is a function which can be expressed in the infinite trigonometric series, then

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Here a_0, a_n and b_n are Fourier coefficients of f(x). Now, if $f(x)$ is integrable in the internal $[-\pi, \pi]$. Then Fourier coefficients are defined as

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

and, $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin n x \, dx$$

Now we discuss these coefficients for even and odd functions .

For even function

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \, dx$$

for $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos n x \, dx$$

as f(x) is an even function and cos nx is also an even function. Therefore, f(x) cos n x being product of two even functions is also an even function.

Hence $$a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx$$

and for $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin n x \, dx$$

Here f(x) sin nx, being product of an even and an odd function, is odd

Therefore $b_n = 0$ (by definition)

For Odd function

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = 0 \qquad \text{(by definition)}$$

for $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos n x \, dx$$

as f(x) cos n x is an odd function, being product of an odd and an even function.

Therefore

$$a_n = 0 \qquad \text{(by definition)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin n\,x\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin n\,x\,dx$$

Because $f(x)$ sin nx is an even function, being the product of both odd functions.

Now if we define the periodic function f(x) in the interval $[0, 2\pi]$, then the coefficient of the Fourier series are given by

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos n\,x\,dx$$

and $$b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin n\,x\,dx$$

11.6 Extended Rule of Integration by parts:

This rule is often used in solving the integral in the problems of Fourier Series. This rule of by parts is applied successively a number of times, until we get a standard form of the integrand.

If f and g are two functions, extended rule will be applied in the following form $\int fg\,dx = f g_1 - f' g_2 + f'' g_3 \ldots + (-1)^n f^{n-1} g_n + (-1)^n \int f^n g_n\,dx$

Where f', f″, f‴, ...are first, second, third derivatives and so on, and g_1, g_2, g_3,.......are first, second, third integrals and so on.

Example 1:

Determine the Fourier series for the following functions.

(i) $f(x) = x^2$, $-\pi \le x \le \pi$

(ii) $f(x) = x^2$, $0 \le x \le 2\pi$

Solution (i):

For the interval $-\pi \le x \le \pi$

We know that x^2 is an even function, therefore $b_n = 0$ and a_0, a_n will be calculated as,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2}{\pi}\frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\ \cos n\,x\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos n\,x\,dx$$

Here we use extended rule of integration by parts

$$= \frac{2}{\pi}\left[\left| x^2\frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right|_0^{\pi}\right]$$

We know that $\sin 0 = 0$ and $\sin \pi = 0$

Therefore $$a_n = \frac{2}{\pi}\left[-2\pi\left(-\frac{\cos n\pi}{n^2}\right) + 2(0)\left(-\frac{\cos n\,0}{n^2}\right)\right]$$

$$= \frac{4}{n^2}\ \mathrm{Cos}\ n\,\pi + 0 = \frac{4}{n^2}\ (-1)^n$$

because for n =1,2, 3……….n, $\Rightarrow \cos n\pi = -1, 1, -1\ldots\ldots = (-1)^n$

Fourier series for the function is given by

$$x^2 = \frac{1}{2}a_0 + \sum_1^{\infty} a_n \cos n\,x + b_n \sin n\,x.$$

$$= \frac{1}{2}\frac{2\pi^2}{3} + \sum_1^{\infty}\left[\frac{4}{n^2}\ (-1)^n\ \cos nx + (0)\sin n\,x\right]$$

$$= \frac{\pi^2}{3} + 4\sum_1^{\infty}(-1)^n\ \frac{\cos n\,x}{n^2}$$

$$= \frac{\pi^2}{3} + 4\left(-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^3} + \ldots\ldots\ldots\right)$$

Solution (ii):

No we solve $f(x) = x^2$ for $0 \leq x \leq 2\pi$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{\pi}\left|\frac{x^3}{3}\right|_0^{2\pi}$$

$$= \frac{1}{\pi}\frac{(2\pi)^3}{3} = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos n\,x\,dx$$

Applying extended rule of integration by parts

$$a_n = \frac{2}{\pi}\left[\left| x^2\frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right|_0^{2\pi}\right]$$

Here $\sin 2\pi = \sin 0 = 0$

Therefore $a_n = \left[\left\{-2(2\pi)\left(-\frac{\cos 2n\pi}{n^2}\right)\right\} - \left\{-2(0)\left(\frac{\cos 0}{n^2}\right)\right\}\right]$

$$= \frac{1}{\pi}\left[+\frac{4\pi \cos 2n\pi}{n^2}\right]$$

$= 4\,\frac{\cos 2n\pi}{n^2} = \frac{4}{n^2}$ because for $n = 1, 2, 3, \ldots \Rightarrow \cos 2n\pi = 1$.

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin n x \, dx$$

Applying extended rule of integration by parts.

$$b_n = \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[4\pi^2\left(-\frac{\cos 2n\pi}{n}\right) + \frac{2\cos 2n\pi}{n^3}\right] - \frac{1}{\pi}\left[\frac{2\cos 0}{n^3}\right]$$

$$= \frac{-4\pi}{n} + \frac{2}{n^3\pi} - \frac{2}{n^3\pi} = \frac{-4\pi}{n}$$

Now Fourier Series for the given function in the given interval is

$$x^2 = \frac{1}{2}\frac{8\pi^2}{3} + \sum_1^\infty\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

$$= \frac{4\pi^2}{3} + 4\sum_1^\infty\left(\frac{\cos nx}{n^2} - \frac{4\pi}{n}\sin n x\right)$$

Example 2: Find the Fourier Series.

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

Solution:

As f(x) is a periodic function of period 2π, therefore its Fourier series is

$$\frac{1}{2}a_0 + \sum_{1}^{\infty}(a_n \cos n x + b_n \sin n x)$$

So here we evaluate a_0, a_n and b_n

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (0)\, dx + \frac{1}{\pi}\int_{0}^{\pi} (1)\, dx$$

$$a_0 = 0 + \frac{1}{\pi}|x|_0^\pi = \frac{1}{\pi}(\pi - 0) = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} 0 \cos n x + \frac{1}{\pi}\int_{0}^{\pi} 1 \cos n x\, dx$$

$$= 0 + \frac{1}{\pi}\left[+\frac{\sin n x}{n}\right]_0^\pi = \frac{1}{n\pi}[\sin n\pi - \sin 0]$$

$$= \frac{1}{\pi}[0 - 0] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} (0) \sin n x\, dx + \frac{1}{\pi}\int_{0}^{\pi} 1 \sin n x\, dx$$

$$= 0 + \frac{1}{\pi}\left[-\frac{\cos n x}{n}\right]_0^\pi = -\frac{1}{n\pi}[\cos n\pi - \cos 0]$$

$$= -\frac{1}{n\pi}\left[(-1)^n - 1\right] = \frac{1}{n\pi}\left[(-1)^{n+1} + 1\right]$$

Fourier series becomes

$$f(x) = \frac{1}{2} + \sum_{1}^{\infty}\left[0 \cos n x + \frac{1}{n\pi}\left\{(-1)^{n+1} + 1\right\} \sin n x\right]$$

$$= \frac{1}{2} + \frac{1}{\pi}\left[\frac{2\sin x}{1} + \frac{2\sin 3x}{3} + \frac{2\sin 5x}{5} + \ldots\ldots\ldots\ldots\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\ldots\ldots\right]$$

Example 3:

Obtain the Fourier Series for e^x in $[-\pi, \pi]$

Solution:

Here a_o, a_n and b_n will be evaluated.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}\left|e^x\right|_{-\pi}^{\pi} = \frac{1}{\pi}(e^{\pi} - e^{-\pi})$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos n\,x\,dx$$

Let $I_1 = \int e^x \cos n\,x\,dx = \int \cos nx . e^x\,dx$

Let $\cos n\,x = 1^{st}$ function and $e^x = 2^{nd}$ function

Now we apply integration by parts

$$I_1 = \cos n\,x \int e^x - \int\left(\frac{d}{dx}\cos nx \int e^x\,dx\right) dx$$

$$= \cos n\,x\,e^x - \int(-n)\sin n\,x\,e^x\,dx$$

$$= \cos n\,x\,e^x + n\left[\sin nx \int e^x\,dx - \int\left(\frac{d}{dx}\sin nx \int e^x\,dx\right) dx\right]$$

$$= \cos n\,x\,e^x + n\left[\sin nx\,e^x - \int n\cos n\,x\,e^x\,dx\right]$$

$$I_1 = \cos n\,x\,e^x + n\sin n\,x\,e^x - n^2 I_1$$

$$I_1 + n^2 I_1 = (\cos n\,x + n\sin n\,x)\,e^x$$

$$I_1(1 + n^2) = (\cos n\,x + n\sin n\,x)\,e^x$$

$$I_1 = \frac{e^x}{(1 + n^2)}(\cos n\,x + n\sin n\,x)$$

$$a_n = \frac{1}{\pi}\left|\frac{e^x}{(1 + n^2)}(\cos nx + n\sin nx)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{(1+n^2)}\cos n\pi - \frac{e^{-\pi}}{(1+n^2)}\cos(-n\pi)\right] \quad \left\{\begin{array}{l}\therefore \sin nx = 0 \\ \text{for } x = \pi \text{ or } -\pi\end{array}\right.$$

$$= \frac{e^{x}}{\pi(1+n^2)}\left[e^{\pi}(-1)^n - e^{-\pi}(-1)^n\right]$$

$$a_n = \frac{(-1)^n}{\pi(1+n^2)}\left[e^{\pi} - e^{-\pi}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x} \text{Sin } nx \, dx$$

Let $I_2 = \int \sin n x . e^{x} dx$

Applying integration by parts by taking

Ist function = sin nx and second functions = e^x

$$I_2 = \sin nx \int e^{x} dx - \int\left(\frac{d}{dx}\sin nx \int e^{x} dx\right) dx$$

$$= \sin nx \, e^{x} - \int n \cos n x \, e^{x} dx$$

$$= \sin nx \, e^{x} - n\left[\cos nx \int e^{x} dx - \int\left(\frac{d}{dx}\cos nx \int e^{x} dx\right) dx\right]$$

$$I_2 = \sin nx \, e^{x} - n\left[\cos n x \, e^{x} - \int -n \sin n x \, e^{x} dx\right]$$

$$= \sin nx \, e^{x} - n \cos nx \, e^{x} - n^2 \int \sin n x \, e^{x} dx]$$

$$I_2 = \sin n x \, e^{x} - n \cos n x \, e^{x} - n^2 I_2$$

$$I_2 + n^2 I_2 = \sin n x \cdot e^{x} - n \cos n x \, e^{x}$$

$$I_2 = \frac{e^{x}}{(1+n^2)}(\sin n x - n \cos n x)$$

$$b_n = \frac{1}{\pi}\left|\frac{e^{x}}{1+n^2}(\sin nx - n \cos nx)\right|_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left\{\frac{ne^{\pi}}{1+n^2}\,[-\cos n\pi]-\frac{ne^{-\pi}}{1+n^2}\,[-\cos(-n\pi)\right\}$$

$$=\frac{1}{\pi}\left[-\frac{ne^{\pi}}{1+n^2}(-1)^n+\frac{ne^{-\pi}}{1+n^2}(-1)^n\right]$$

$$=\frac{n(-1)^n}{\pi(1+n^2)}\,\left[-e^{\pi}+e^{-\pi}\right]$$

Now Fourier Series

$$f(x)=\frac{1}{2}\,a_0+\sum_{1}^{\infty}(\,a_n\cos nx+b_n\sin nx) \qquad \text{becomes}$$

$$f(x)=\frac{1}{2\pi}(e^{\pi}-e^{-\pi})+\sum_{1}^{\infty}\left[\frac{(-1)^n}{\pi(1+n^2)}(e^{\pi}-e^{-\pi})\cos nx+\frac{n(-1)^n}{\pi(1+n)^2}(-e^{\pi}+e^{-\pi})\sin nx\right]$$

$$=\frac{1}{2\pi}(e^{\pi}-e^{-\pi})+\frac{e^{\pi}-e^{-\pi}}{\pi}\sum_{1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos nx-n\sin nx)$$

$$=\frac{1}{\pi}(e^{\pi}-e^{-\pi})\left\{\frac{1}{2}+\sum_{1}^{\infty}\frac{(-1)^n}{1+n^2}\;(\cos nx-n\sin nx)\right\}$$

Example 4: Find the Fourier Series for

$$f(x)=\begin{cases}-1, & -\pi<x<0\\ 0, & x=0\\ 1, & 0<x<\pi\end{cases}$$

Solution: $a_0=\frac{1}{\pi}\int_{-\pi}^{0}(-1)dx+0+\frac{1}{\pi}\int_{0}^{\pi}(1)\;dx$

$$a_0=\frac{1}{\pi}\left(\left|-x\right|_{-\pi}^{0}+\left|x\right|_{0}^{\pi}\right)$$

$$= \frac{1}{\pi}(-0-\pi) + \frac{1}{\pi}(\pi-0) = -1+1 = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} -1.\cos n\,x\,dx + 0 + \frac{1}{\pi}\int_{0}^{\pi} 1.\cos nx\,dx$$

$$= -\frac{1}{\pi}\left|\frac{\sin nx}{n}\right|_{-\pi}^{0} + \frac{1}{\pi}\left|\frac{\sin nx}{n}\right|_{0}^{\pi}$$

$$= -\frac{1}{\pi}(0) + \frac{1}{\pi}(0) = 0$$

Now $b_n = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin nx\,dx + 0 + \frac{1}{\pi}\int_{0}^{\pi} 1 \sin n\,x\,dx$

$$= \frac{1}{\pi}\left|\frac{\cos nx}{n}\right|_{-\pi}^{0} + \frac{1}{\pi}\left|-\frac{\cos nx}{n}\right|_{0}^{\pi}$$

$$= \frac{1}{n\pi}[1-\cos(-n\pi)] + \frac{1}{n\pi}(-\cos n\pi + \cos 0)$$

$\therefore \quad \text{Cos}(-n\pi) = \cos n\pi$

$$= \frac{1}{n\pi}(1-(-1)^n) + \frac{1}{n\pi}[-(-1)^n + 1]$$

$$b_n = \frac{2}{n\pi}(1-(-1)^n) = \frac{2}{n\pi}[1+(-1)^{n+1}]$$

So Fourier series becomes

$$f(x) = \frac{1}{2}(0) + \sum_{1}^{\infty}\left[0\cos nx + \frac{2}{n\pi}\{1+(-1)^{n+1}\}\sin nx\right]$$

$$= 0 + \sum_{1}^{\infty}\frac{1}{n\pi}[2+2(-1)^{n+1}]\sin nx$$

$$= \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$$

Example 5:

Find the Fourier series for

$f(x) = \sin x \quad , \quad -\pi \le x \le \pi$

Solution:

We know that sin x is an odd function and its coefficient for Fourier series will be $a_0 = a_n = 0$ and

b_n will be evaluated

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x \sin nx\,dx$$

$$= -\frac{2}{\pi}\int_0^{\pi} -\sin x \sin nx\,dx = -\frac{1}{\pi}\int_0^{\pi} -2\sin x \sin nx\,dx$$

$$= -\frac{1}{\pi}\int_0^{\pi}\left[\cos(1+n)x - \cos(1-n)x\right]dx$$

$$= -\frac{1}{\pi}\left[\frac{\sin((1+n))x}{(1+n)} - \frac{\sin(1-n)x}{(1-n)}\right]_0^{\pi} = 0$$

For all values of n except n=1

$\because \sin 0 = 0$ and $\sin \pi = 0$

For n = 1

$$b_1 = \frac{2}{\pi}\int_0^{\pi}\sin^2 x\,dx = \frac{2}{2\pi}\int_0^{\pi}(1-\cos 2x)\,dx$$

$$= \frac{1}{\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{\pi}(\pi - 0) = 1$$

So this series becomes

$$f(x) = \frac{1}{2}(0) + \sum_1^{\infty}\left(0\cos nx + 1\sin nx\right) = \sum_1^{\infty} \text{Sin } nx$$

$$= \text{Sin } x + \text{Sin } 2x + \text{Sin } 3x + \ldots\ldots\ldots\ldots$$

Exercise 11

Q.1: Expand the following functions in Fourier Series.

(i) $f(x) = x, \quad -\pi \le x \le \pi$

(ii) $f(x) = x.\sin x \quad -\pi \le x \le \pi$

(iii) $f(x) = e^{2ax} \quad -\pi \le x \le \pi$

Q.2: Expand the following

(i) $$f(x)=\begin{cases}1, & 0\le x\le \pi\\ 2, & \pi\le x\le 2\pi\end{cases}$$

(ii) $$f(x)=\begin{cases}\dfrac{-\pi}{4}, & -\pi<x<0\\ \dfrac{\pi}{4}, & 0<x\le\pi\end{cases}$$

(iii) $$f(x)=\begin{cases}0, & -\pi<x<0\\ 1, & 0<x<\dfrac{\pi}{2}\\ 0, & \dfrac{\pi}{2}<x<\pi\end{cases}$$ Period 2π

Q.3: Expand the following functions in the Fourier Series.

$f(x)=|x|$, $-\pi\le x\le \pi$

Also deduce that $\dfrac{\pi^2}{8}=1+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\dfrac{1}{7^2}+\dots\dots\dots\dots$

Q.4: Expand $f(x)=x+x^2$ in Fourier Series in $-\pi\le x\le\pi$

Also deduce that $\dfrac{\pi}{4}+\dfrac{\pi^2}{6}=\sum_{1}^{\propto}\dfrac{1}{n^2}$

Q.5: Obtain Fourier Series for $f(x)=\cos\alpha x$, $-\pi<x\le\pi$.

Also deduce that $\cot\alpha\pi=\dfrac{1}{\pi}\left(\dfrac{1}{\alpha}-\sum_{1}^{\propto}\dfrac{2\alpha}{n^2-\alpha^2}\right)$, Here α is not an integer

Q.6: Obtain Fourier Series for $f(x)=\dfrac{\pi}{4}$, $0<x<\pi$

Also prove that: $\dfrac{\pi}{4}=\dfrac{\sin x}{1}+\dfrac{\sin 3x}{3}+\dfrac{\sin 5x}{5}$

Answers 11

Q.1: (i) $2\left\{\dfrac{\sin x}{1}+\dfrac{\sin 2x}{2}+\dfrac{\sin 3x}{3}-\dots\dots\dots\right\}$

(ii) $1+2\sum_{n=2}^{\infty}\dfrac{(-1)^n}{n^2-1}\cos nx$

(iii) $$\frac{e^{2ax}+e^{-2ax}}{\pi}\left\{\frac{1}{4a}+\sum_{1}^{\infty}\frac{(-1)n}{4a^2+n^2}(2a\cos nx-n\sin nx)\right\}$$

Q.2: (i) $$\frac{3}{2}-\frac{2}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+\ldots\ldots\right]$$

(ii) $$\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{3}+\ldots\ldots.$$

(iii) $$\frac{1}{4}+\frac{1}{\pi}\left[\frac{\cos x}{1}+\frac{\sin x}{1}-\frac{\cos 3x}{3}+\frac{\sin 3x}{3}+\ldots\ldots\right]$$
$$+\frac{2}{\pi}\left[\frac{\sin 2x}{2}+\frac{\sin 6x}{6}+\ldots\ldots\right]$$

Q.3: $$f(x)=\frac{1}{2}-\frac{4}{\pi^2}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}----\right)$$

Deduction for $x = 0, f(x) = 0 \Rightarrow \frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots..$

Q.4: $$f(x)=x+x^2=\frac{\pi^2}{3}+\sum_{1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx+\frac{2}{n}(-1)^{n+1}\sin nx$$

Deduction: Put $x=\pi$,

$$\frac{\pi}{4}+\frac{\pi^2}{6}=\sum_{1}^{\infty}\frac{1}{n^2}$$

Q.5: $$f(x)=\cot\alpha x=\frac{\sin\alpha x}{\alpha\pi}+\sum_{1}^{\infty}\frac{2(-1)^n\alpha\sin\alpha\pi\cos nx}{\pi(\alpha^2-n^2)}$$

Deduction: Put $x=\pi$

$$\cot\alpha\pi=\frac{1}{\pi}\left(\frac{1}{\alpha}-\sum_{1}^{\infty}\frac{2\alpha}{n^2-\alpha^2}\right)$$

Q.6: $$f(x)=\frac{\pi}{4}=\frac{\pi}{8}+\sum_{1}^{\infty}\frac{1}{2n}\sin nx$$

Deduction: $$\frac{\pi}{2}=\frac{\pi}{4}+\sum_{1}^{\infty}\frac{1}{n}\sin nx$$
$$\frac{\pi}{4}=\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}$$

Solved Short Questions

Write the short answers of the following :

Q.1: Define Fourier Series.

Q.2: What is Fourier Series?

Q.3: If a function is odd integrable on $[-\overline{\wedge},\overline{\wedge}]$ then which co-efficient exist.

Q.4: If a function is even integrable on $[-\overline{\wedge},\overline{\wedge}]$ then which co-efficient exist.

Q.5: What are Fourier Co-efficient?

Q.6: If x^2 is integrable in $[-\overline{\wedge},\overline{\wedge}]$, then which of the Fourier Co-efficient will non-Zero.

Q.7: If x is integrable in $[-\overline{\wedge},\overline{\wedge}]$, then which of the Fourier co-efficient will be Zero.

Q.8: Write down formula for extended rule of integration.

Answers

Q1: A Fourier Series decomposes a periodic function into sum of a set of simple oscillating functions, called Sines & Cosines

Q2: The infinite sum $\frac{a_o}{2} + \sum_{n=1}^{\propto} (a_n \text{Cos } n x + b_n \text{ Sin } nx)$ is called Fourier Series.

Q3: Only b_n exists and a_o, $a_n = 0$

Q4: a_o, a_n exists and $b_n = 0$

Q5: Constant a_o, a_n and b_n present in the Fourier Series are called Fourier co-efficient.

Q6. a_o, a_n are non- zero

Q7. a_o, a_n are zero

Q8. $\int fg\,dx = f g_1 - f' g_2 + f'' g_3 \dots + (-1)^n f^{n-1} g_n + (-1)^n \int f^n g_n\,dx$

Objective Type Questions

Q.1: Encircle the correct one of the given answers in each item.

1. The series $\frac{a_0}{2} + \sum_{n=1}^{\propto} (a_n \, Cos \, n \, x + b_n \, Sin \, nx)$ is

(a) Bionomial

[✓ (b) Fourier]

(c) Arithmetic

(d) Geometric

2. If a function $f(-x) = -f(x)$, then function is

(a) even

[✓ (b) odd]

(c) linear

(d) constant

3. If a function $f(-x) = f(x)$, then function is

[✓ (a) even]

(b) odd

(c) linear

(d) constant

4. In an odd function, the Fourier co- efficient a_0 is

[✓ (a) zero]

(b) 1

(c) −1

(d) 2

5. In an odd function, the Fourier co- efficient a_n is

[✓ (a) zero]

(b) 1

(c) −1

(d) 2

6. In an even function, the Fourier co- efficient b_n is

[✓ (a) zero]

(b) 1

(c) −1

(d) 2

7. The period of sin x is

(a) π

[✓ (b) 2π]

(c) −π

(d) −2π

8. The period of cos x is

(a) π

[✓ (b) 2π]

(c) −π

(d) −2π

9. If a function f(x) is periodic if $f(x) = f(\text{-----------})$

[✓ (a) $x \pm T$]

(b) $\pm \frac{x}{T}$

(c) $\pm x T$

(d) None of these

10 If function f(x) is even then $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$

(a) $\int_{0}^{\pi} f(x)\,dx$ (b) $\frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

(c) $\frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$ (d) $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

Answers

Q1.

1.	b	**2.**	b	**3.**	a	**4.**	a	**5.**	a
6.	a	**7.**	b	**8.**	b	**9.**	a	**10.**	c

Chapter 12

Laplace Transformation

12.1 Introduction:

We now present a systematic and elegant procedure that is widely used in circuit analysis and in the study of feedback and control. The theory takes its form from a symbolic method developed by the English engineer Oliver Heaviside. The modern approach to this method is based on the Laplace transformation. It enables one to solve many problems without going to the trouble of finding the general solution and then evaluating the arbitrary constants. The procedure can be extended to systems of equations to partial differential equations of electrical network and mechanical oscillations and to integral equations and it often yields results more readily than other techniques.

12.2 Laplace transformation:

The basis of this method is the transformation defined by

$$\overline{f}(s) = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt = L\ \{f(t)\}\ldots\ldots\ldots\ldots\ldots(1)$$

The function F(s) is the Laplace transform of f(t), and the operator L that transforms f(t) into F(s) is the Laplace transform operator. The functional relation expressed between F(s) and f(t) is written in the form F(s) = L {f(t)}.

It should be emphasized that equation (1) describes the action of L, not only on f(t) but on any function to which L can be applied Thus,

$$L\{U(t)\} = \int_0^\infty U(t)\, e^{-st}\, dt,\quad L\ \{V(t)\} = \int_0^\infty V(t)\, e^{-st}\, dt \text{ and so on}$$

Example 1:

Let $f(t) = 1$ when $t \geq 0$, Find $\overline{f}(s)$ (Laplace transform)

Solution: Since $f(t) = 1$

$$\therefore\quad \overline{f}(s) = L\ \{f(t)\} = \int_0^\infty f(t)\, e^{-st}.\, dt$$

$$\overline{f(s)} = L\{f(t)\} = \int_0^\infty 1.e^{-st}\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}\left[e^{-\infty} - e^0\right]$$

$$= -\frac{1}{s}[0-1] \qquad \therefore \qquad e^{-\infty} = 0$$

$$= \frac{1}{s}$$

Example 2:

Let $f(t) = t$ when $t \geq 0$, Find $L\{f(t)\}$

Solution : Since $f(t) = t$

$$\therefore \qquad \overline{f(s)} = L\{f(t)\} = \int_0^\infty f(t)\,e^{-st}.dt$$

$$\overline{f(s)} = L\{f(t)\} = \int_0^\infty t.e^{-st}\,dt$$

Integration by parts

$$= \left[t.\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\,dt$$

$$= \quad 0 + \frac{1}{s}\int_0^\infty e^{-st}.\,dt$$

$$= \quad \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s^2}\left[e^{-\infty} - e^0\right]$$

$$= \quad -\frac{1}{s^2}[0-1] = \frac{1}{s^2}$$

Example 3:

Let $f(t) = e^{at}$ When $t \geq 0$, and a is constant, Find $L\{f(t)\}$

Solution: Since $f(t) = e^{at}$

$$\therefore \quad \overline{f(s)} = L\,\{f(t)\} = \int_0^\infty f(t)\, e^{-st}.\,dt$$

$$L\,\{e^{at}\} = \int_0^\infty e^{-st}.\, e^{at}\,dt = \int_0^\infty e^{-(s-a).t}\,dt$$

$$= \left| \frac{e^{-(s-a).t}}{-(s-a)} \right|_0^\infty$$

$$= -\frac{1}{s-a}\left[e^{-\infty} - e^{0}\right] = -\frac{1}{s-a}\left[0-1\right]$$

$$= \frac{1}{s-a} \quad ; \quad s > a$$

12.3 Properties of Laplace Transformation :

Linearity :

1. $$L\,\{c\,f(t)\} = \int_0^\infty c\,f(t)\, e^{-st}\,dt$$

$$= c\int_0^\infty f(t)\, e^{-st}\,dt$$

$$L\,\{c\,f(t)\} = c\,L\,\{f(t)\}$$

2. $$L\,\{\alpha f(t) + \beta\, g(t)\} = \int_0^\infty \{\alpha f(t) + \beta\, g(t)\, e^{-st}\,dt\}$$

$$= \int_0^\infty \alpha f(t) e^{-st}\,dt + \int_0^\infty \beta\, g(t)\, e^{-st}\,dt$$

$$= \int_0^\infty \alpha f(t) e^{-st}\,dt + \beta\int_0^\infty g(t)\, e^{-st}\,dt$$

$$L\,\{\alpha\, f(t) + \beta\, g(t) = \alpha L\, f(t) + \beta\, L\, g(t)$$

Which shows that the operator L is linear.

Laplace transform of derivatives :

3. $$L\,[f'(t)] = \int_0^\infty e^{-st}.\, f'(t)\,dt$$

Integrating by parts

$$= \left|(e^{-st}) f(t)\right|_{0}^{\infty} + s\int_{0}^{\infty} e^{-st}.\, f(t)\, dt$$

$$= [e^{-\infty} f(\infty) - e^{0} f(0)] + s\, L f(t)$$

$$= -f(0) + s\, L\, (f(t)$$

$L[f'(t)] = s\, L\, f(t) - f(0)$

4. $L[f''(t)] = s\, L\, [f'(t)] - f'(0)$

$$= s\, [s\, Lf(t)] - f(0)] - f'(0)$$

$L[f''(t)] = s^2 L\, [f(t)] - s\, f(0) - f'(0)$

5. $L[f'''(t)] = s^2 L\, [f'(t)] - s\, f'(0) - f''(0)$

$$= s^2[s\, L\, f(t) - f(0)] - s\, f'(0) - f''(0)$$

$$= s^3\, L\, f(t) - s^2\, f(0) - s\, f'(0) - f''(0)$$

Repetition of this process gives.

$L[f^{m}(t)] = s^{m} L\, f(t) - s^{m-1} f(0) - s^{m-2} f_1(0) \ldots - s\, f_{m-2}(0) - f_{m-1}(0)$

Where, $f_1(0) = f'(0), \ldots\ldots, f_{m-1}(0) = f^{(m-1)}(0)$

Example 4: Find $L(t^n)$

Solution: Let $f(t) = t^n \Rightarrow f(0) = 0$

$f_1(t) = n\, t^{n-1} \Rightarrow f_1(0) = 0$

$f_2(t) = n(n-1)\, t^{n-2} \Rightarrow f_2(0) = 0$

$f_{n-1}(t) = n(n-2) \ldots\ldots 2.t \Rightarrow f_{n-1}(0) = 0$

$f_n(t) = f^{n}(t) = n\,(n-1)\,(n-2) \ldots\ldots 2.1 = n!$

By property 4

$L\,\{f^{n}(t)\} = S^{n} L\, f(t) - S^{h-1} f(0) - S^{n-2} f_1(0) - \ldots\ldots f_{n-1}(0)$

$L\{n!\} = S^{n} L\,(t^{n})$

$n!\, L\{1\} = S^{n} L\,(t^{n})$

$n!\, (\frac{1}{s}) = S^{n} L\,(t^{n})$

$$L(t^n) = \frac{n!}{s^{n+1}}$$

Example 5:

Find $L(t^3)$

Solution: **Method – I**

Since $L(t^n) = \frac{n!}{s^{n+1}}$

Since $L(t^3) = \frac{3!}{s^{3+1}} \quad = \frac{6}{S^4}$

Method – 2:

Let $f(t) = t^3$, then $f(0) = 0$

$f'(t) = 3t^2$, $f(0) = 0$

$f''(t) = 6t$, $f''(0) = 0$

$f'''(t) = 6$

By the formula (property – 5)

$$Lf'''(t) = s^3 L f(t) - s^2 f(0) - s f'(0) - f''(0)$$

$$L\{6\} = s^3 L f(t) - s^2(0) - s(0) - 0$$

$$6L\{1\} = s^3 L f(t)$$

$$6\left(\frac{1}{s}\right) = s^3 L(f(t))$$

$$\therefore \quad L(t^3) = \frac{6}{s^4}$$

Example 6:

(i) Find $L[\sin \omega t]$ (ii) $L[\cos \omega t]$

Solution: **Method –I**

Since by example 2

$$L(e^{at}) = \frac{1}{S-a}$$

Put $a = i\omega$

$$L(e^{i\omega t}) = \frac{1}{S-i\omega}$$

$$L(\cos \omega t + i \sin \omega t) = \frac{1}{S-i\omega} \times \frac{S+i\omega}{S+i\omega} =$$

$$L(\cos\omega t) + i\,L(\sin\omega t) = \frac{S}{S^2+\omega^2} + i\,\frac{\omega}{S^2+\omega^2}$$

Comparing the real and imaginary parts we get

(i) $L[\cos\omega t] = \dfrac{S}{S^2+\omega^2}$ (Real part)

(ii) $L[\sin\omega t] = \dfrac{\omega}{S^2+\omega^2}$ (Imaginary part)

Method – 2: (i) Find $L[\sin\omega t]$

Here, $f(t) = \sin\omega t$, $f(0) = \sin 0 = 0$

$f'(t) = \omega\cos\omega t$, $f'(0) = \omega\cos 0 = \omega$

$f''(t) = -\omega^2\sin\omega t$

Since, $L[f''(t)] = S^2L[f(t)] - S\,f(0) - f'(0)$

$$L[-\omega^2\sin\omega t] = S^2L[\sin\omega t] - 0 - \omega$$

$$-\omega^2L[\sin\omega t] - S^2L[\sin\omega t] = -\omega$$

$$(S^2+\omega^2)\,L[\sin\omega t] = \omega$$

$$L[\sin\omega t] = \frac{\omega}{S^2+\omega^2}$$

(ii) Find $L[\cos\omega t]$

Here, $f(t) = \cos\omega t$, $f(0) = \cos 0 = 1$

$f'(t) = -\omega\sin\omega t$, $f'(0) = -\omega\sin 0 = 0$

$f''(t) = -\omega^2\cos\omega t$

Since, $L[f''(t)] = S^2L[f(t)] - S\,f(0) - f'(0)$

$$L[f''(t)] = S^2L[\cos\omega t] - S\,f(0) - f'(0)$$

$$L[-\omega^2\cos\omega t] = S^2L[\cos\omega t] - S - 0$$

$$(S^2+\omega^2)\,L[\cos\omega t] = S$$

$$L[\cos\omega t] = \frac{S}{S^2+\omega^2}$$

Property 6: **Laplace transform of the derivative of order n :**

Let f(t) be continuous function for $t \geq 0$, then the Laplace transforms of nth order derivative is

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left[L\{f(t)\}\right]$$

Proof:

Since $\overline{f(s)} = L\{f(t)\} = \int_0^\infty e^{-st}.\, f(t)\, dt$

Differentiating w.r.t. s

$$\frac{d}{ds}\left[\overline{f(s)}\right] = \int_0^\infty -t.\, e^{-st}.\, f(t)\, dt$$

$$\frac{d}{ds}\left[\overline{f(s)}\right] = (-1)\int_0^\infty t.\, e^{-st}.\, f(t)\, dt$$

$$(-1)\frac{d}{ds}\left[\overline{f(s)}\right] = L\{t\, f(t)\}$$

On differentiating once again

$$(-1)\frac{d^2}{ds^2}\left[\overline{f(s)}\right] = \int_0^\infty t.e^{-st} f(t)\,(-t)$$

$$(-1)\frac{d^2}{ds^2}\left[\overline{f(s)}\right] = (-1)\int_0^\infty t^2\, e^{-st} f(t)\, dt$$

$$\therefore \quad (-1)^2\frac{d^2}{ds^2}\left[\overline{f(s)}\right] = L\{t^2 f(t)\}$$

Similarly

$$(-1)^n\frac{d^n}{ds^n}\left[\overline{f(s)}\right] = L\{t^n f(t)\}$$

Example 7:

Let $f(t) = t^n$ for all $t \geq 0$, find $L\{t^n\}$

Solution:

$$L\{t^n f(t)\} = (-1)^n\frac{d^n}{ds^n}\left[\overline{f(s)}\right] = (-1)^n \frac{d^n}{ds^n}\left[L\{f(t)\}\right]$$

Here, $f(t) = 1$

$$L\{t^n.1\} = (-1)^n \frac{d^n}{ds^n}\left[\frac{1}{s}\right] \qquad \therefore \qquad L\{1\} = \frac{1}{s}$$

$$= (-1)^n \frac{(-1)^n n!}{s^{n+1}}$$

$$= \frac{n!}{s^{n+1}}$$

12.4 Inverse Laplace Transforms:

The operation by which we recover f(t) from $L\{f(t)\} = F(s)$ is called inverse Laplace Transform and is denoted by L^{-1}.

Thus if $L\{f(t)\} = F(s)$

Then $f(t) = L^{-1}[F(s)]$

Table of Laplace Transforms and Inverse Laplace Transforms

Sr#	f(t)	L f(t)	L {f(t)}	$f(t) = L^{-1}[F(s)]$
1	1	$^1/_s$	$^1/_s$	1
2	t	$^1/_{s^2}$	$^1/_{s^2}$	t
3	t^n	$\frac{n!}{S^{n+1}}$	$\frac{n!}{S^{n+1}}$	t^n
4	e^{ct}	$\frac{1}{S-C}$	$\frac{1}{S-C}$	e^{ct}
5	e^{-ct}	$\frac{1}{S+C}$	$\frac{1}{S+C}$	e^{-ct}
6	$\sin \omega t$	$\frac{\omega}{S^2+\omega^2}$	$\frac{\omega}{S^2+\omega^2}$	$\sin \omega t$
7	$\cos \omega t$	$\frac{S}{S^2+\omega^2}$	$\frac{S}{S^2+\omega^2}$	$\cos \omega t$
8	$t\,e^{-ct}$	$\frac{1}{(S+C)^2}$	$\frac{1}{(S+C)^2}$	$t\,e^{-ct}$
9	$\frac{t^{n-1}\,e^{-ct}}{(n-1)!}$ n is + ve	$\frac{1}{(S+C)^n}$	$\frac{1}{(S+C)^n}$	$\frac{t^{n-1}\,e^{-ct}}{(n-1)!}$
10	$e^{-ct}(1-ct)$	$\frac{S}{(S+C)^2}$	$\frac{S}{(S+C)^2}$	$e^{-ct}(1-ct)$

Example 8:

Find $L^{-1}[f(s)]$ of $\dfrac{s}{(s-a)(s-b)}$, $a \neq b$

Solution:

$$L^{-1}\left\{\frac{s}{(s-a)(s-b)}\right\} = L^{-1}\left[\frac{1}{a-b}\left\{\frac{a}{s-a} - \frac{b}{s-b}\right\}\right] \text{ (by partial fraction)}$$

$$= \frac{1}{a-b}\left[L^{-1}\left(\frac{a}{s-a}\right) - L^{-1}\left(\frac{b}{s-b}\right)\right]$$

$$= \frac{1}{a-b}\left(a\,e^{at} - b\,e^{bt}\right)$$

Note:

$$\therefore \quad L^{-1}\left(\frac{1}{S-a}\right) = e^{at}$$

$$L^{-1}\left(\frac{1}{s-b}\right) = e^{bt}$$

Exercise 12

1. Find the Laplace transforms of the following functions.
   (i) 4 (ii) $3t$ (iii) t^2
   (iv) at (v) $t^2 - 2t$ (vi) e^{-at}
2. Find the Laplace transforms of the following functions.
   (i) $3t + 4$ (ii) $t^2 + at + b$ (iii) $\cos 3t$
   (iv) $\sin 4t$ (v) $a \cos 2t$ (vi) $\sin t \,.\, \cos t$
3. Show that Laplace transforms of

   $$L\{t^2 f'(t)\} = S\,\overline{f''(s)} + 2\,\overline{f'(s)}$$
4. Prove that

   (i) $L\{e^{at}\cos \omega t\} = \dfrac{s-a}{(s-a)^2+\omega^2}$

   (ii) $L\{e^{at}\sin \omega t\} = \dfrac{\omega}{(s-a)^2+\omega^2}$
5. Let $f(t) = 2 \sin \omega t$. Find $L\{f(t)\}$
6. Find inverse Laplace transforms of the following

(i) $\frac{5}{s-3}$ (ii) $\frac{1}{s^2+25}$ (iii) $\frac{9}{s^2+3S}$

(iv) $\frac{1}{s(s+1)}$ (v) $\frac{1}{s(s^2+1)}$ (vi) $\frac{1}{(s+1)(s-2)}$

(vii) $\frac{1}{(s+a)(s+b)}$

Answers

Q.1: (i) $\frac{4}{s}$ (ii) $\frac{3}{s^2}$ (iii) $\frac{2!}{s^2}$

(iv) $\frac{a}{s^2}$ (v) $\frac{2!}{s^3}-\frac{2}{s^2}$ (vi) $\frac{1}{s+a}$

Q.2: (i) $\frac{3}{S^2}+\frac{4}{S}$ (ii) $\frac{2}{s^3}+\frac{a}{s^2}+\frac{b}{s}$ (iii) $\frac{s}{s^2+3^2}$

(iv) $\frac{4}{s^2+4^2}$ (v) $\frac{a\,s}{s^2+2^2}$ (vi) $\frac{1}{s^2+4}$

Q.5: $\frac{2\omega}{s^2+\omega^2}$

Q.6: (i) $5e^{3t}$ (ii) $\frac{1}{5}\sin 5t$ (iii) $3-3e^{3t}$

(iv) $1-e^{3t}$ (v) $1-\cos t$ (vi) $\frac{1}{2}[e^{-t}-e^{2t}]$

(vii) $\frac{1}{(a-b)}[-e^{-at}+e^{-bt}]$

Solved Short Questions

Write the short answers of the following :

Q.1: Define Laplace Transformation.

Q.2: What is the main use of Laplace Transformation?

Q.3: Find Laplace transformation of a constant K.

Q.4: What is Laplace transformation of t^n?

Q.5: Prove that $L\{u'(t)\} = S\,L\,\{u(t)\} - u(0)$

Q.6: Write the formula for $L\{u''(t)\}$

Q.7: Write Laplace transformation of e^{at}.

Q.8: If $L\{t^n\} = \frac{n!}{S^{n+1}}$ then what will be $L\{t^7\}$

Q.9: Write Laplace transformation of $t\,e^{at}$

Q.10: Find Laplace transformation of t.

Q.11: What is Laplace transformation of Sin (7t)?

Q.12: If $L\{e^{at}\} = \frac{1}{S - a}$ then what will be the Laplace transformation of (i) e^{-4t} (ii) $e^{t/2}$

Q.13: What is Laplace transformation of Cos (6t)?

Q.14: Define inverse Laplace transformation.

Q.15: What is the most important method to find the inverse Laplace transformation of function?

Q.16: What is the inverse Laplace transformation of (i) $\frac{1}{S^n}$ (ii) $\frac{1}{s + a}$?

Q.17: What is inverse Laplace transformation of the function $\frac{4}{S^2 + 16}$?

Q.18: Find $L^{-1}\left\{\frac{1}{s - a} - \frac{1}{s + a}\right\}$

Q.19: What is inverse Laplace transformation of $\frac{2}{S^3}$?

Answers

Q1. $$L\,\{U(t)\} = \int_0^\infty e^{-st}\,u\,(t)\,dt$$

Q2. The Laplace transformation is used to solve differential equations. It is more useful than classical methods.

Q3. $\frac{K}{S}$

Q4. $\frac{n!}{S^{n+1}}$ Q6. $L\{u''(t)\} = S^2 L\{u(t)\} - S\,u(o) - u'(o)$

Q7. $\frac{1}{s-a}$ Q8. $\frac{5040}{S^8}$ Q9. $\frac{1}{(S-a)^2}$

Q10. $\frac{1}{S^2}$ Q11. $\frac{7}{S^2+49}$

Q12. (i) $\frac{1}{S+4}$ (ii) $\frac{2}{2S-1}$ Q13. $\frac{S}{S^2+36}$

Q14. u(t) is called inverse Laplace transformation of F(S)and written as,
$u(t) = L^{-1}\{F(S\}$

Q15. The partial fraction method is most important method to find inverse Laplace transformation.

Q16. (i) $\frac{t^{n-1}}{(n-1)!}$ (ii) e^{-at}

Q17. Sin (4t) Q18. $e^{at} - e^{-at}$ Q19. t^2

Objective Types Questions

Q.1: Encircle the correct one of the given answers in each item.

1--- L {f(t)} (the Laplace Transform of f(t)) is

[✓ (a) $\int_0^\infty f(t)\, e^{-st}.\,dt$] (b) $\int_0^\infty f(t)\, e^{st}.\,dt$

(c) $\int_0^\infty f(t)\, e^{t}.\,dt$ (d) $\int_0^\infty f(t)\, e^{s}.\,dt$

2--- Laplace transform of the function f(x) = 1 is

(a) $1/s^3$ (b) $1/s^2$ [✓ (c) $1/s$] (d) $-1/s$

3--- Laplace transform of the function f(x) = t is

(a) $1/s$ [✓ (b) $1/s^2$] (c) $1/s^3$ (d) $-1/s$

4--- Laplace transform of the function f(x) = 2 is

(a) $s/2$ (b) $2/s^2$ (c) $2/s^3$ [✓ (d) $2/s$]

5--- Laplace transform of the function $f(x) = e^t$ is

[✓ (a) $\frac{1}{s-1}$] (b) $\frac{1}{s}$ (c) $s - 1$ (d) s

6--- Laplace transform of the function $f(x) = e^{at}$ is

(a) $\frac{1}{s+a}$ [✓ (b) $\frac{1}{s-a}$] (c) $s - a$ (d) $s + a$

7--- Laplace transform of the function $f(x) = e^{-at}$ is

(a) $\frac{1}{s+a}$ [✓ (b) $\frac{1}{s-a}$] (c) $s - a$ (d) $s + a$

8--- The Laplace Transform of f(t) = sinωt

[✓ (a) $\frac{\omega}{S^2+\omega^2}$] (b) $\frac{1}{s^2+\omega^2}$

(c) $\frac{S}{S^2+\omega^2}$ (d) $\frac{1}{s^2-\omega^2}$

S. P ✓

9--- The Laplace Transform of $f(t) = t^n$

(a) $\frac{n}{s^{n+1}}$ [✓ (b) $\frac{n!}{s^{n+1}}$]

(c) $\frac{1}{s^{n+1}}$ (d) $\frac{n!}{s^n}$

10--- The inverse Laplace transform $L^{-1}\left(\frac{1}{s}\right)$ is equal to

[✓ (a) 1] (b) 2 (c) 3 (d) 4

11--- The inverse Laplace transform $L^{-1}\left(\frac{1}{s^2}\right)$ is equal to

(a) 1 [✓ (b) t] (c) t^2 (d) t/2

12--- $\frac{S}{S^2+\omega^2}$ is inverse Laplace transform of

(a) Sin ωt [✓ (b) cos ωt] (c) $\sin\frac{t}{\omega}$ (d) $\cos\frac{t}{\omega}$

13--- $\frac{\omega}{S^2+\omega^2}$ is inverse Laplace transform of

[✓ (a) Sin ωt] (b) cos ωt (c) $\sin\frac{t}{\omega}$ (d) $\cos\frac{t}{\omega}$

S.P ✓

14--- $\frac{1}{s^2+1}$ is inverse Laplace transform of

[✓ (a) Sin t] (b) cos t (c) $\sin\left(\frac{1}{t}\right)$ (d) $\cos\left(\frac{1}{t}\right)$

15--- $\frac{s}{s^2+1}$ is inverse Laplace transform of

(a) Sin t [✓ (b) cos t] (c) $\sin\left(\frac{1}{t}\right)$ (d) $\cos\left(\frac{1}{t}\right)$

16--- $L^{-1}\left(\frac{1}{s-1}\right)$ is equal to

(a) e^{-t} (b) e^{2t} (c) $\frac{1}{t}$ [✓ (d) e^{t}]

17--- $L^{-1}\left(\frac{1}{s+1}\right)$ is equal to

(a) e^{t} (b) e^{2t} (c) $\frac{1}{t}$ [✓ (d) e^{-t}]

Answers

Q1.

1.	a	2	c	3.	b	4.	d	5.	a
6.	b	7.	a	8.	a	9	b	10.	a
11.	b	12.	b	13.	a	14.	a	15.	b
16.	d	17.	d						

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